Hadronic matrix elements of gluon operators
in the instanton vacuum

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Abstract

We propose a method to evaluate hadronic matrix elements of QCD gluon operators in the instanton vacuum. We construct the ground state of the interacting instanton ensemble for non-zero ϑ–angle using a variational principle. A method to study the ϑ–dependence of observables on the lattice is suggested. We then derive the effective fermion action, which allows to calculate hadronic correlation functions in a $1/N_c$–expansion (Nambu–Jona-Lasinio type effective fermion theory). Gluon operators are systematically represented as effective fermion operators. Physical matrix elements are obtained after integrating the correlation functions over fluctuations of the numbers of instantons. The influence of the fermion determinant on the topological susceptibility is taken into account. Our effective description gives matrix elements fully consistent with the trace and $U(1)_A$ anomalies. The approach allows to consistently evaluate the nucleon matrix elements of various gluon and mixed quark–gluon operators in a chiral soliton picture of the nucleon.

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Contents

1 Introduction

It is now widely recognized that instantons play an important, if not crucial, role in the dynamics of strong interactions. After their discovery twenty years ago [1], their relevance to hadron phenomenology was explored in a number of pioneering papers [2, 3, 4], while a quantitative theory of the instanton medium was still lacking. A treatment of the interacting instanton ensemble by means of the Feynman variational principle showed how the instanton medium stabilizes itself at rather low densities [5]. In particular, the ratio of the average size of the instantons to the average separation between nearest neighbors, $\bar{\rho}/\bar{R}$, was computed as $\simeq 1/3$, the value suggested from phenomenological considerations [4]. The instanton vacuum was also studied in direct lattice experiments, by the so-called cooling procedure[1] [7, 8, 9, 10, 11, 12]. It was observed that instantons and antiinstantons (I's and $\bar{I}$'s for short) are the only non-perturbative gluon configurations surviving after a sufficient smearing of the quantum gluon fluctuations. The measured properties of the $I\bar{I}$ medium, such as the ratio $\bar{\rho}/\bar{R}$, appear to be close to those computed from the variational principle [9, 11, 12].

The success of the instanton vacuum lies in its explanation of all phenomena related to the dynamical breaking of chiral symmetry in QCD. The mechanism of chiral symmetry breaking is the delocalization of the would-be fermion zero modes of individual instantons [13, 14], leading to a finite fermion spectral density at eigenvalue zero, proportional to the chiral condensate, $\langle \bar{\psi}\psi \rangle$ [15]. Equivalently, one observes that quarks propagating in the instanton medium acquire a dynamical momentum–dependent mass [14, 16]. A massless pion appears as a collective excitation.

It should be noted that instantons do not lead to confinement [3]. However, it has been realized in many ways that it is chiral symmetry breaking, and not confinement, which determines the basic characteristics of light hadrons (except, probably, for highly excited states) [4, 14, 17, 18]. This point of view has recently received direct support from lattice measurements of correlation functions in the cooled vacuum [11]. Cooling the quantum fluctuations eliminates not only the perturbative one–gluon exchange, but also the linear confining potential. Nevertheless, the correlation functions of various hadronic currents in the cooled vacuum, where only I's and $\bar{I}$'s are left, appear to be quite similar to those of the true "hot" vacuum.

Detailed numerical studies of correlation functions of meson and baryon currents in the instanton medium by Shuryak, Verbaarschot and Schäfer have shown impressive agreement with phenomenology [19, 20]. Correlation functions in the instanton vacuum can also be computed analytically [14, 17]. By integrating first over the instanton coordinates, one derives an effective fermion action of the form of a Nambu–Jona-Lasinio model [17]. It exhibits a many–fermion interaction with a specific spin–flavor structure, first suggested by 't Hooft [2]. The effective fermion theory can be studied systematically in a $1/N_c$–expansion. In this formulation, chiral symmetry is broken by the instanton–induced many–fermionic interactions. Correlation functions of meson currents can be computed as fermion loop diagrams with a momentum cutoff defined by the average instanton size, $\bar{\rho}^{-1} \simeq 600\,\mathrm{MeV}$. The study of correlation functions of baryon currents at large N_c leads to a picture of baryons as chiral solitons — N_c valence quarks moving in a self–consistent

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[1]For a more complete list of references, see [6].

meson field [21]. This approach to low–energy baryon structure has been worked out extensively and gives a good account of the static properties of non–strange and strange baryons [22]. Furthermore, by calculating correlation functions of baryon currents with various weak and electromagnetic current operators, baryon form factors have been computed [23], which are found to be in excellent agreement with experiment.

In the description of the deep–inelastic structure of the nucleon, one faces the task of evaluating nucleon matrix elements of certain operators containing gluon fields [24]. There are essentially two types of such gluonic operators. Operators of twist 2 are related to the moments of the nucleon structure functions. Operators of higher twist, such as $F_{\mu\nu}^2$, $F_{\mu\nu}\widetilde{F}_{\mu\nu}$, or $\bar{\psi}\widetilde{F}_{\mu\nu}\gamma_\nu\psi$ arise in the description of power corrections. Their evaluation is a challenge for any model of the QCD vacuum capable of describing the nucleon. Of particular interest is the nucleon matrix element of $F\widetilde{F}$, the subject of the so–called proton spin problem[2] [24]. The nucleon matrix elements have to be extracted from correlation functions of nucleon currents with the respective gluon operators. A related problem, which also requires evaluation of matrix elements of operators involving fermion and gluon fields, is the estimation of vacuum condensates of "mixed" operators like $\bar{\psi}F_{\mu\nu}\sigma_{\mu\nu}\psi$, which give corrections in QCD sum rules [26].

The object of this paper is the study of hadronic matrix elements of gluon operators in the instanton vacuum. Our framework is the grand canonical ensemble of interacting I's and $\bar{I}$'s, to which fermions are coupled mainly through the zero modes. The fluctuations of the number of I's (N_+) and $\bar{I}$'s (N_-), are essential to realize the renormalization properties of QCD, such as the trace anomaly and the $U(1)_A$–anomaly. Our strategy will be as follows. We first consider the instanton ensemble in gluodynamics and derive the distribution of the sizes and numbers of pseudoparticles from a variational principle [5]. With this instanton size distribution, we determine the effective fermion action in quenched approximation, for a fixed number of instantons, $N_\pm$. Correlation functions are then evaluated in a two–step procedure. In the first step, we average over instanton coordinates and the fermion field for fixed $N_\pm$. For correlation functions of operators consisting of fermion fields (such as meson or baryon currents), this average can be performed directly with the help of the effective fermion action. Moreover, as we shall show, even correlation functions with gluon operators can be reduced to averages in the effective fermion theory: in the course of integrating over instanton coordinates, the QCD gluon operator is replaced by an effective many–fermion operator. In the final step, we then pass from canonical to grand canonical averages, by integrating over fluctuations of the number of I's and $\bar{I}$'s, $N_\pm$. The probability for these fluctuations is given by the instanton partition function including the fermion determinant. By this procedure, we can calculate correlation functions of nucleon currents with gluon operators, and extract the nucleon matrix elements by taking appropriate large–time limits.

Our intention here is twofold. First, we want to set up a framework for concrete calculations. All correlation functions are systematically reduced to averages in a Nambu–Jona-Lasinio type effective fermion theory, which can be evaluated using established techniques [22]. Second, we want to demonstrate that the fundamental renormalization properties of QCD are preserved in this approach, even at the level of hadronic matrix elements.

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[2]For a recent lattice estimate of the nucleon matrix element of the topological charge, see [25].

For this reason, we shall focus on the operators $F_{\mu\nu}^2$ and $F_{\mu\nu}\widetilde{F}_{\mu\nu}$. In QCD, the nucleon matrix element of $F_{\mu\nu}^2$ is uniquely fixed by the trace anomaly, while the matrix element of $F_{\mu\nu}\widetilde{F}_{\mu\nu}$ is related to the isoscalar axial charge by the $U(1)_A$ anomaly. In the instanton formulation, the values of these operators are related, respectively, to $N_+ + N_-$ and $N_+ - N_-$. We shall show that the nucleon matrix elements calculated according to our prescription are obtained in agreement with the QCD anomalies. The proper inclusion of non–topological and topological fluctuations of the numbers of instantons will be crucial for this property.

Matrix elements of gluon operators depend in general on the renormalization point. When computed in the instanton vacuum, they are obtained at the scale set by the instanton background, $1/\bar{\rho}$. In general, the results must be subjected to QCD evolution, to make contact with experimental data at higher momentum scales. This problem will be addressed elsewhere. Here, we concentrate on gluon operators related to QCD anomalies, about which exact statements can be made in QCD at the hadronic scale.

The plan of this paper is as follows. In section 2, we consider the grand canonical ensemble of instantons in gluodynamics. The distribution of the total number of instantons, $N_+ + N_-$, is derived from the QCD trace anomaly. We discuss the dependence of physical quantities on the ϑ–angle and suggest a way to measure this dependence on the lattice. We then perform a variational estimate of the grand canonical partition function at finite ϑ–angle, allowing for a general form of average pseudoparticle interaction with different strengths between same and opposite–kind instantons. In section 3, we include fermions in the partition function. We derive the effective fermion action in zero mode approximation, for an ensemble with a fixed number of instantons, $N_\pm$, but allowing for $N_+ \neq N_-$. We outline the bosonization of the 't Hooft interaction in the case of more than one fermion flavor ($N_f > 1$). We then discuss the suppression of the topological susceptibility due to light fermions in QCD on the basis of the $U(1)_A$ and chiral Ward identities. We show that the fermion determinant obtained in the instanton vacuum for $N_+ \neq N_-$ precisely describes this suppression. In section 4, we consider correlation functions of hadronic currents with gluon operators. We show how fixed–$N_\pm$ correlation functions can be represented as averages over the effective fermion theory, in which the gluon operator is replaced by an effective fermion operator. We formulate the rules how to "translate" an arbitrary gluon operator into an effective fermion operator, and demonstrate their consistency. The correlation functions of the grand canonical ensemble are then obtained by averaging over fluctuations of $N_+ + N_-$ and $N_+ - N_-$. It is shown that the basic renormalization properties of QCD are preserved in this framework: we verify the low–energy theorem of scale invariance for correlation functions with the operator $\int d^4 x \, F_{\mu\nu}^2$, and the realization of the $U(1)_A$ anomaly for the operator $\int d^4 x \, F_{\mu\nu}\widetilde{F}_{\mu\nu}$. In section 5, we then derive the nucleon matrix elements of $F_{\mu\nu}^2$ and $F_{\mu\nu}\widetilde{F}_{\mu\nu}$ in our approach. The nucleon matrix element of $F_{\mu\nu}^2$ is obtained consistent with the trace anomaly. The matrix elements of $F_{\mu\nu}\widetilde{F}_{\mu\nu}$ reduces to the isosinglet axial coupling constant, $g_A^{(0)}$, as calculated within the effective fermion theory. This allows us to consistently interpret the result of a former chiral soliton calculation of $g_A^{(0)}$ as an estimate of the nucleon matrix element of $F_{\mu\nu}\widetilde{F}_{\mu\nu}$ in the large–N_c limit [27]. Conclusions and an outlook are presented in section 6.

2 Statistical mechanics of instantons

2.1 The grand partition function of instantons

The essence of the instanton vacuum is that the number of "pseudoparticles" is not fixed, hence one is dealing with a *grand* canonical ensemble. Let us denote the number of I's and $\bar{I}$'s, respectively, by N_+, N_-. The grand partition function is generically given as

$$Z(\mu, \vartheta) = \sum_{N_+, N_-} \exp\left((\mu + i\vartheta)N_+\right) \exp\left((\mu - i\vartheta)N_-\right) Z_{N_\pm}, \tag{2.1}$$

$$Z_{N_\pm} = \frac{1}{N_+! N_-!} \prod_I^{N_+ + N_-} \int d^4 z_I \, d\rho_I \, dU_I \, d_0(\rho_I) \, \exp(-U_{\text{int}}). \tag{2.2}$$

Here, the integrals go over the collective coordinates of instantons: their center coordinates, z_I, sizes, ρ_I, and orientations, given by $SU(N_c)$ unitary matrices in the adjoint representation, U_I; dU_I means the Haar measure normalized to unity. It is useful to introduce complex chemical potentials for I's and $\bar{I}$'s, $\mu \pm i\vartheta$, where ϑ is the usual vacuum angle. The instanton interaction potential, U_{int}, depends on the separation between pseudoparticles, $z_I - z_J$, their sizes, ρ_I, ρ_J, and their relative orientation, $U_I U_J^\dagger$. By $d_0(\rho)$ we denote the one–instanton weight. In the one–loop approximation it is given by [2, 28]

$$d_0(\rho) = \frac{C_{N_c}}{\rho^5} \beta(M_{\text{cut}})^{2N_c} \exp\left[-\beta(\rho)\right], \tag{2.3}$$

where $\beta(\rho)$ is the one–loop inverse charge,

$$\beta(\rho) = \frac{8\pi^2}{g^2(\rho)} = b \log\left(\frac{1}{\Lambda\rho}\right), \qquad b = \frac{11}{3} N_c. \tag{2.4}$$

Note that β in the pre-exponential factor of eq.(2.3) starts to "run" only at the two–loop level, hence its argument is taken at the ultra-violet cut-off, M_{cut}. The coefficient C_{N_c} depends on the renormalization scheme; in the Pauli–Villars scheme it is

$$C_{N_c} = \frac{4.60 \, e^{-1.68 N_c}}{\pi^2 (N_c - 1)! (N_c - 2)!}. \tag{2.5}$$

The relation between the QCD scale parameters, Λ, in different schemes is

$$\Lambda_{\text{PV}} = 1.09 \, \Lambda_{\overline{\text{MS}}} = 31.32 \, \Lambda_{\text{lattice}}. \tag{2.6}$$

If the scheme is changed, one has to change the coefficient according to $C_{N_c} \rightarrow C'_{N_c} = C_{N_c}(\Lambda/\Lambda')^b$.

In the two–loop approximation, the instanton weight is given by [5, 29]

$$d_0(\rho) = \frac{C_{N_c}}{\rho^5} \beta(\rho)^{2N_c} \exp\left[-\beta_{\text{II}}(\rho) + \left(2N_c - \frac{b'}{2b}\right)\frac{b'}{2b}\frac{\log\beta(\rho)}{\beta(\rho)} + O(\beta(\rho)^{-1})\right], \tag{2.7}$$

where $\beta_{\text{II}}(\rho)$ is the inverse charge to two–loop accuracy,

$$\beta_{\text{II}}(\rho) = \beta(\rho) + \frac{b'}{2b} \log \frac{2\beta(\rho)}{b}, \qquad b' = \frac{34}{3} N_c^2. \tag{2.8}$$

The partition function, eq.(2.1), is actually normalized to the perturbative partition function (without instantons). Therefore, when the four–dimensional volume, V, goes to infinity, which we always assume, the logarithm of Z gives the ground state (vacuum) energy density, θ_{44}, with that of perturbation theory subtracted,

$$Z = \exp\left[-V\left(\theta_{44} - (\theta_{44})_{\text{pert}}\right)\right]. \tag{2.9}$$

Here, $\theta_{\mu\nu}$ is the energy–momentum tensor. If the vacuum state is isotropic, one has $\theta_{44} = \frac{1}{4}\theta_{\mu\mu}$. Using the trace anomaly,

$$\theta_{\mu\mu} = \frac{\beta(g^2)}{4g^4}F_{\mu\nu}^2 \simeq -\frac{b}{32\pi^2}F_{\mu\nu}^2, \tag{2.10}$$

where $F_{\mu\nu}^2 \equiv F_{\mu\nu}^a F_{\mu\nu}^a$ is the gluon field strength squared and $\beta(g^2)$ is the Gell-Mann–Low function (not to be confused with the inverse charge $\beta(\rho)$),

$$\beta(g^2) \equiv \frac{dg^2(M_{\text{cut}})}{d\log M_{\text{cut}}} = -b\frac{g^4}{8\pi^2} - \frac{b'}{2}\frac{g^6}{(8\pi^2)^2} - \cdots, \tag{2.11}$$

one gets [5]

$$Z = \exp\left(V\frac{b}{4}\frac{1}{32\pi^2}\left\langle F_{\mu\nu}^2\right\rangle_{\text{np}}\right). \tag{2.12}$$

Here, $\langle F_{\mu\nu}^2\rangle_{\text{np}}$ is the gluon field vacuum expectation value, which is due to non-perturbative fluctuations, i.e. the gluon condensate [30]. It is important that this is a renormalization group–invariant quantity[3], meaning that its dependence on the ultraviolet cutoff, M_{cut}, and the bare charge given at this cutoff, $g^2(M_{\text{cut}})$, is such that it is actually cutoff–independent,

$$\frac{1}{32\pi^2}\left\langle F_{\mu\nu}^2\right\rangle_{\text{np}} = c\left[M_{\text{cut}}\exp\left(-\int_{\cdots}^{g^2(M_{\text{cut}})}\frac{dg^2}{\beta(g^2)}\right)\right]^4 \simeq c'M_{\text{cut}}^4\exp\left[-\frac{32\pi^2}{bg^2(M_{\text{cut}})}\right]. \tag{2.13}$$

(Here, c, c' are constants independent of M_{cut}.) By definition of the gluodynamics partition function,

$$Z_{\text{glue}} = \int \mathcal{D}A_\mu \exp\left(-\frac{1}{4g^2(M_{\text{cut}})}\int d^4x\, F_{\mu\nu}^2\right), \tag{2.14}$$

the l.h.s. of eq.(2.13) is equal to

$$\frac{1}{32\pi^2}\left\langle F_{\mu\nu}^2\right\rangle_{\text{np}} = -\frac{1}{V}\frac{1}{8\pi^2}\frac{d\log Z_{\text{glue}}}{d[1/g^2(M_{\text{cut}})]}. \tag{2.15}$$

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[3]To be more precise, the RNG–invariant quantity is $\langle\theta_{\mu\mu}\rangle$, see eq.(2.10). However, if the coupling constant at the ultra-violet cutoff scale, $g^2(M_{\text{cut}})$, is small enough, it is sufficient to use the beta function to one–loop order.

Applying the same differentiation to eq.(2.13) once more, one gets a low-energy theorem [30],

$$
\frac{1}{(8\pi^2)^2}\frac{d^2 \log Z_{\text{glue}}}{d[1/g^2(M_{\text{cut}})]^2} = \frac{1}{(32\pi^2)^2}\left\langle \int d^4x\, F_{\mu\nu}^2 \int d^4y F_{\mu\nu}^2 \right\rangle_{\text{np}} - \frac{1}{(32\pi^2)^2}\left\langle \int d^4x\, F_{\mu\nu}^2 \right\rangle_{\text{np}}^2
$$

$$
= \frac{4}{b}\frac{1}{32\pi^2}\left\langle \int d^4x\, F_{\mu\nu}^2 \right\rangle_{\text{np}}. \tag{2.16}
$$

If the bare coupling, $g^2(M_{\text{cut}})$, is not chosen small enough, there are obvious corrections to this formula, following from the higher-order terms in the beta function, eq.(2.11).

This low-energy theorem has important consequence for instantons: it predicts the dispersion of the number of pseudoparticles in the grand canonical ensemble [5]. Assuming the instanton ensemble to be sufficiently dilute, and taking into account that the one–instanton action is

$$
\left(\int d^4x\, F_{\mu\nu}^2 \right)_{1-\text{inst}} = 32\pi^2, \tag{2.17}
$$

one can rewrite the low-energy theorem, eq.(2.16), as

$$
\langle N^2 \rangle - \langle N \rangle^2 = \frac{4}{b}\langle N \rangle = \frac{12}{11\,N_c}\langle N \rangle, \tag{2.18}
$$

where $N \equiv N_+ + N_-$ is the total number of pseudoparticles. Thus, it follows directly from the renormalization properties of the Yang–Mills theory that the dispersion of the number of instantons is less than for a free gas of instantons, for which one would get a Poisson distribution, $\langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$. At $N_c \to \infty$, the dispersion becomes zero.

One concludes that the interaction of instantons with each other is crucial to support the necessary renormalization properties of the underlying theory: any cutoff of the integrals over instanton sizes "by itself", such as a cutoff due to a possible infrared fixed point of the beta function, as suggested recently by Shuryak [31], leads to the Poisson distribution and hence to the violation of the low energy theorem, eq.(2.18).

By further differentiating $\log Z_{\text{glue}}$ with respect to the bare coupling, $g^2(M_{\text{cut}})$, one can easily generalize eq.(2.18) to higher moments of the distribution. For example,

$$
\langle N^3 \rangle - 3\langle N^2 \rangle\langle N \rangle + 2\langle N \rangle^3 = \left(\frac{4}{b}\right)^2 \langle N \rangle, \tag{2.19}
$$

etc.. Summarizing these formulas, one concludes that the distribution of the number of pseudoparticles should, for large $\langle N \rangle$, be given by

$$
P(N) \propto \exp\left[-\frac{b}{4}N\left(\log\frac{N}{\langle N \rangle} - 1 \right) \right]. \tag{2.20}
$$

A convenient way to generate all moments of the number distribution is to use the chemical potential, μ, introduced in eq.(2.1). Renormalizability predicts that the grand partition function, eq.(2.1), as a function of μ, should behave as

$$
Z(\mu) = \exp\left[\frac{b}{4}\langle N \rangle \exp\left(\frac{4\mu}{b} \right) \right]. \tag{2.21}
$$

Differentiating $\log Z(\mu)$ with respect to μ and setting $\mu = 0$, one obtains all the moments in accordance with the low-energy theorems.

2.2 ϑ–angle dependence

The statements made in the previous section are independent of the ϑ–angle and should thus hold at any ϑ. This means that in eqs.(2.20, 2.21) only the quantity $\langle N \rangle$ may depend on ϑ. Hence the partition function can be written in the form

$$Z(\mu, \vartheta) \;=\; \exp\left[\frac{b}{4} \langle N \rangle_0 \, f(\vartheta) \, \exp\left(\frac{4\mu}{b} \right) \right], \qquad f(0) \;=\; 1, \qquad (2.22)$$

where $\langle N \rangle_0$ is the average total number of pseudoparticles at $\vartheta = 0$. The function $f(\vartheta)$ can not be determined from general considerations. Nevertheless, comparing eq.(2.22) and eq.(2.1), one realizes that eq.(2.22) imposes a non-trivial restriction on the instanton interactions. The factorization of the μ– and ϑ–dependence would be a property of non-interacting particles, but in this case the factors $b/4$ would be replaced by unity.

Information on $f(\vartheta)$ can be extracted by differentiating $\log Z$ with respect to ϑ. The second derivative is proportional to the topological susceptibility of the vacuum,

$$
\begin{aligned}
\langle Q_t^2 \rangle &\equiv \left\langle \frac{1}{32\pi^2} \int d^4 x \, F\tilde{F} \frac{1}{32\pi^2} \int d^4 y \, F\tilde{F} \right\rangle \;=\; -\frac{\partial^2 \log Z}{\partial \vartheta^2} \\
&= \langle (N_+ - N_-)^2 \rangle \;=\; -\frac{b}{4} \langle N \rangle_0 \frac{\partial^2 f(\vartheta)}{\partial \vartheta^2}.
\end{aligned}
\qquad (2.23)
$$

The dependence of the QCD observables on the ϑ angle is of fundamental importance. However, until now no direct lattice measurements with non-zero ϑ have been performed. The reason is that the inclusion of a non-zero ϑ–term in lattice simulations is ambiguous and, moreover, makes the integration measure complex. The above discussion suggests an unambigious method to study the ϑ–dependence of observables on the lattice. Indeed, cooling down a gluon configuration, one finds — and that rather early — the topological charge of the configuration, $Q_t = \Delta \equiv N_+ - N_-$, which is the canonical conjugate to ϑ. If $P(\Delta)$ is the fraction of configurations with given Δ, the probability of a configuration with given ϑ is given by the Fourier transform,

$$Z(\vartheta) \;=\; \sum_\Delta P(\Delta) \, \exp(i\vartheta \, \Delta). \qquad (2.24)$$

Measuring a QCD observable, O, such as the string tension *etc.*, for a set of configurations, and knowing the topological charge of those configurations, one can establish the dependence of the vacuum average of the observable on ϑ,

$$O(\vartheta) \;=\; \sum_\Delta O(\Delta) P(\Delta) \, \exp(i\vartheta \Delta). \qquad (2.25)$$

The topological charge of a configuration, $\Delta = N_+ - N_-$, can rather easily be established after cooling. The total number of I's and $\bar{I}$'s, $N = N_+ + N_-$, is a less well–defined quantity, as it tends to decrease during the lattice cooling. The question arises: When to stop the cooling procedure? When does one hit the true instanton vacuum in the lattice simulation? The answer is provided by the dispersion of N, eq.(2.18), or, more generally, by eqs.(2.20, 2.21), which differ from the Poisson distribution by factors of $b/4 = 11 \, N_c/12$.

When that distribution in the total number of pseudoparticles is achieved, one can measure various observables like the condensates, correlation functions, *etc.*. The most non-trivial check, however, that one is dealing with the true instanton vacuum, is the abovementioned factorization of the μ– and ϑ–dependence in the logarithm of the partition function. Let the probability of a configuration with N_+ I's and N_- $\bar{I}$'s be $P(N_+, N_-)$. If the original gluon configurations are cooled sufficiently, but not over-cooled, the grand partition function should have the form of eq.(2.22),

$$
\begin{aligned}
Z(\mu, \vartheta) &= \sum_{N_\pm} P(N_+, N_-) \, \exp[\mu(N_+ + N_-)] \, \exp[i\vartheta(N_+ - N_-)] \\
&= \exp\left[\frac{b}{4}\langle N\rangle_0 \, f(\vartheta) \, \exp\left(\frac{4\mu}{b}\right)\right].
\end{aligned} \tag{2.26}
$$

Once such non-trivial behavior in μ and ϑ is established, one can read off the gluon condensate as

$$
\frac{1}{32\pi^2} \left\langle F_{\mu\nu}^2 \right\rangle_{\mathrm{np}} = \frac{\langle N\rangle_0}{V}. \tag{2.27}
$$

The above statements are about all one can say on the dispersion of pseudoparticles on general grounds. To obtain more information on the instanton vacuum, one has to go into dynamics, *i.e.*, take into account the instanton–instanton interaction in a quantitative way. In the next section, we estimate the ground state of the grand canonical partition function, eq.(2.1), using a variational principle.

2.3   Variational estimate of the grand partition function

In this section, we show that actually very limited information on the instanton interactions is needed to make a variational estimate of the basic properties of the instanton ensemble. We shall neglect the many-body forces, an approximation which is justified *a posteriori*, if the medium turns out to be sufficiently dilute.

An intuitively clear description is achieved by introducing the actual size distributions of instantons in the medium, which do not necessarily coincide with the weight of an isolated instanton, $d_0(\rho)$, of eqs.(2.3, 2.7). Moreover, the results of section 2.1 show that they necessarily do not coincide with $d_0(\rho)$ — otherwise the renormalization properties of the theory would be violated. Following ref.[5], we now derive the "best" distribution functions, $d_\pm(\rho)$, from a variational principle. This approach preserves all the general requirements discussed in sections 2.1 and 2.2.

As an intermediate step, it will be convenient to work with the canonical ensemble. We first derive the best $d_\pm(\rho)$ for a given number of I's and $\bar{I}$'s, and leave the summation over $N_\pm$ for the end. Let us write the partition function for given $N_\pm$, eq.(2.2), in the form

$$
Z_{N_\pm} = \frac{1}{N_+! N_-!} \int \prod_I^{N_+ + N_-} d\xi_I \exp[-E(\xi)], \qquad E(\xi) = -\sum_I^{N_+ + N_-} \log d_0(\rho_I) + U_{\mathrm{int}}(\xi). \tag{2.28}
$$

Here, $\xi_I = (z_I, \rho_I, U_I)$ denote the set of collective coordinates of the Ith pseudoparticle, and ξ those of all $N_+ + N_-$ pseudoparticles.

For the purpose of a variational estimate, we introduce a "non-interacting" partition function,

$$
\begin{aligned}
Z_1 &= \frac{1}{N_+! N_-!} \int \prod_I^{N_+} d\xi_I \, d_+(\rho_I) \prod_{\bar{I}}^{N_-} d\xi_{\bar{I}} \, d_-(\rho_{\bar{I}}) = \frac{1}{N_+! N_-!} \int \prod_I^{N_+ + N_-} d\xi_I \exp[-E_1(\xi)], \\
E_1(\xi) &= -\sum_I^{N_+} \log d_+(\rho_I) - \sum_{\bar{I}}^{N_-} \log d_-(\rho_{\bar{I}}).
\end{aligned}
\tag{2.29}
$$

Here, $d_\pm(\rho)$ are the effective size distributions in the ensemble of interacting N_+ I's and N_- $\bar{I}$'s. Note that these distributions, naturally, depend on $N_\pm$. The partition function, eq.(2.29), is immediately expressed through fugacities, $\zeta_\pm$,

$$
Z_1 = \frac{1}{N_+! N_-!} (V\zeta_+)^{N_+} (V\zeta_-)^{N_-}, \qquad \zeta_\pm = \int d\rho \, d_\pm(\rho).
\tag{2.30}
$$

To find the best $d_\pm(\rho)$ for a given interaction potential, U_{int}, we use the Feynman variational principle, which is based on the convexity of the exponential function. It states that

$$
Z_{N_\pm} = Z_1 \overline{\exp[-(E - E_1)]} \geq Z_1 \exp[-\overline{(E - E_1)}].
\tag{2.31}
$$

Here the averaging is performed with the modified partition function, Z_1,

$$
\overline{A(\xi)} \equiv \frac{1}{Z_1} \frac{1}{N_+! N_-!} \int \prod_I^{N_+} d\xi_I \, d_+(\rho_I) \prod_{\bar{I}}^{N_-} d\xi_{\bar{I}} \, d_-(\rho_{\bar{I}}) \, A(\xi).
\tag{2.32}
$$

In our case we have [5]

$$
\begin{aligned}
\overline{E - E_1} &= \sum_{\epsilon = \pm} \frac{N_\epsilon}{\zeta_\epsilon} \int d\rho \log \frac{d_\epsilon(\rho)}{d_0(\rho)} \\
&+ \frac{1}{2V^2} \sum_{\epsilon_1, \epsilon_2 = \pm} \frac{N_{\epsilon_1}}{\zeta_{\epsilon_1}} \frac{N_{\epsilon_2}}{\zeta_{\epsilon_2}} \int d\xi_1 d\xi_2 \, d_{\epsilon_1}(\rho_1) \, d_{\epsilon_2}(\rho_2) \, U_{\text{int}}(\xi_1, \xi_2).
\end{aligned}
\tag{2.33}
$$

The best effective one–particle distributions, $d_\pm(\rho)$, are those which maximize the r.h.s. of the inequality eq.(2.31), *i.e.*, those which reproduce best the true partition function. We see from eq.(2.33), that in this approach the optimum distribution depends only on averages of the interaction potential over separations and orientations. There are two basic quantities, the average interaction between instantons of the same and of opposite kind; different average interaction between II and $\bar{I}\bar{I}$ are prohibited by CP invariance. Let us denote the basic averages by

$$
\int dU_1 \, dU_2 \, d^4 z_1 \, d^4 z_2 \, U_{\text{int}}^{I\bar{I} \, (\bar{I}I)}(z_1 - z_2, \rho_1, \rho_2, U_1 U_2^\dagger) = V\beta \rho_1^2 \rho_2^2 \gamma_a,
\tag{2.34}
$$

$$
\int dU_1 \, dU_2 \, d^4 z_1 \, d^4 z_2 \, U_{\text{int}}^{II \, (\bar{I}\bar{I})}(z_1 - z_2, \rho_1, \rho_2, U_1 U_2^\dagger) = V\beta \rho_1^2 \rho_2^2 \gamma_s.
\tag{2.35}
$$

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Some comments are in order here. The factors V and β on the r.h.s. of eqs.(2.34, 2.35) are trivial consequences of the fact that the potential depends only on the separation, and that it is proportional to the inverse charge, β. Less trivial is, which argument one should one choose for the running β. Following the reasoning of [5], we are inclined to write $\beta(\langle R \rangle)$, where $\langle R \rangle$ is the average separation, to be determined below. Fortunately, this question is not very important quantitatively, as the dependence on the argument is only logarithmic (see also the discussion of the numerical values below). The product of $\rho_1^2 \rho_2^2$ arises because of dimensions, and because the interaction has to vanish at zero sizes. Finally, we have introduced here two phenomenological constants for the average interaction of pseudoparticles of the same and opposite kind, γ_s and γ_a. It should be mentioned that for the sum ansatz one has [5]

$$\gamma_a \;=\; \gamma_s \;=\; \frac{27}{4}\pi^2 \frac{N_c}{N_c^2 - 1}. \tag{2.36}$$

However, here we would like to take a more liberal stand and allow arbitrary γ_s, γ_a. Nevertheless, the behavior of γ_s, γ_a at large N_c as $1/N_c$, shown by eq.(2.36), is probably a general statement — it arises due to averaging over relative orientations in eqs.(2.34, 2.35).

Substituting the definitions eqs.(2.34, 2.35) into eqs.(2.30, 2.31, 2.33), we get

$$Z_{N_\pm} \;\geq\; \frac{1}{N_+! N_-!} \exp\left(\sum_{\epsilon=\pm} N_\epsilon \left[\log(V\zeta_\epsilon) - \overline{\log \frac{d_\epsilon(\rho)}{d_0(\rho)}} \right] \right)$$
$$\times \exp\left(-\frac{\beta}{2V} \sum_{\epsilon_1,\epsilon_2=\pm} N_{\epsilon_1} N_{\epsilon_2} \overline{\rho_{\epsilon_1}^2 \rho_{\epsilon_2}^2} \gamma_{\epsilon_1 \epsilon_2} \right). \tag{2.37}$$

Here, we have introduced the average sizes,

$$\overline{\rho_\pm^2} \;=\; \frac{1}{\zeta_\pm} \int d\rho\, d_\pm(\rho) \rho^2. \tag{2.38}$$

Varying eq.(2.37) with respect to $d_\pm(\rho)$ we find the best effective size distribution, which takes into account the interactions with the medium,

$$d_\pm(\rho) \;=\; C_\pm\, d_0(\rho)\, \exp(-\rho^2 \alpha_\pm), \tag{2.39}$$
$$\alpha_+ \;=\; \frac{\beta}{V}\left(\gamma_s \overline{\rho_+^2} N_+ + \gamma_a \overline{\rho_-^2} N_- \right), \qquad \alpha_- \;=\; \frac{\beta}{V}\left(\gamma_a \overline{\rho_+^2} N_+ + \gamma_s \overline{\rho_-^2} N_- \right). \tag{2.40}$$

The coefficients $C_\pm$ cancel in eq.(2.37) and can be both put to unity.

We see that the one–instanton size distributions are multiplied by gaussian cutoff functions, whose slope depends on the density of particles. Substituting eq.(2.39) into eqs.(2.30, 2.38) one obtains the fugacities, $\zeta_\pm$, and the average sizes of the instantons, $\overline{\rho_\pm^2}$. In this way, we get the best variational approximation to the fixed–$N_\pm$ partition function,

$$Z_{N_\pm} \;\geq\; \frac{1}{N_+! N_-!} \exp\left(\sum_{\epsilon=\pm} N_\epsilon \log(V\zeta_\epsilon) + \frac{\beta}{2V} \sum_{\epsilon_1,\epsilon_2=\pm} N_{\epsilon_1} N_{\epsilon_2} \overline{\rho_{\epsilon_1}^2 \rho_{\epsilon_2}^2} \gamma_{\epsilon_1 \epsilon_2} \right). \tag{2.41}$$

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Taking the one–loop expression for $d_0(\rho)$, eq.(2.3), one finds

$$\zeta_\pm \;=\; \int d\rho\, d_0(\rho) \exp(-\rho^2 \alpha_\pm) \;=\; \frac{1}{2} C_{N_c} \tilde\beta^{2N_c} \Lambda^b \Gamma(\nu) \alpha_\pm^{-\nu}, \tag{2.42}$$

$$\overline{\rho_\pm^2} \;=\; -\frac{\partial \log \zeta_\pm}{\partial \alpha_\pm} \;=\; \frac{\nu}{\alpha_\pm}, \qquad \nu \;=\; \frac{b-4}{2}, \qquad \tilde\beta \;=\; \beta(M_{\text{cut}}). \tag{2.43}$$

We get a system of equations relating the average sizes of I's and $\bar I$'s to their densities,

$$\overline{\rho_+^2} \left(\gamma_s \overline{\rho_+^2}\, \frac{N_+}{V} \;+\; \gamma_a \overline{\rho_-^2}\, \frac{N_-}{V} \right) \;=\; \frac{\nu}{\beta},$$

$$\overline{\rho_-^2} \left(\gamma_a \overline{\rho_+^2}\, \frac{N_+}{V} \;+\; \gamma_s \overline{\rho_-^2}\, \frac{N_-}{V} \right) \;=\; \frac{\nu}{\beta}. \tag{2.44}$$

At $N_+ = N_- = N/2$, which will be the case at $\vartheta = 0$, when the vacuum is CP symmetric, one has the packing fraction,

$$\left(\overline{\rho_+^2} \right)^2 \frac{N}{V} \;=\; \left(\overline{\rho_-^2} \right)^2 \frac{N}{V} \;=\; \frac{\nu}{\beta} \frac{2}{\gamma_s + \gamma_a}. \tag{2.45}$$

To get the variational estimate of the fixed–$N_\pm$ partition function, $Z_{N_\pm}$, one has to substitute $\overline{\rho_\pm^2}$ from eq.(2.44) into eq.(2.41). Our aim is, however, the grand partition function, $Z(\mu, \vartheta)$, which is obtained from $Z_{N_\pm}$ according to eq.(2.1). The summation over $N_\pm$ in eq.(2.1) can, of course, be done by the steepest descent method. The saddle point values of N_+ and N_- are in general complex and conjugate to each other, and both are proportional to the four–dimensional volume, $V \to \infty$. At $\vartheta = 0$, the saddle point values of $N_\pm$ coincide, and are real. After some straightforward algebra we obtain the logarithm of the grand partition function,

$$\log Z(\mu, \vartheta) \;=\; \frac{b}{4} \langle N \rangle_0 \, \exp\left(\frac{4\mu}{b} \right) f(\vartheta). \tag{2.46}$$

Here, $\langle N \rangle_0$, the average total number of I's and $\bar I$'s at $\vartheta = 0$ and $\mu = 0$, is given by

$$\langle N \rangle_0 \;=\; V \Lambda^4 \left[C_{N_c} \tilde\beta^{2N_c} \Gamma\left(\frac{b-4}{2} \right) \right]^{\frac{4}{b}} \left[\beta \frac{b-4}{2} \frac{\gamma_s + \gamma_a}{2} \right]^{\frac{4-b}{b}}. \tag{2.47}$$

Note that we have obtained the grand partition function exactly in the form predicted in section 2 by considering the renormalization properties of the Yang–Mills theory. Here, we get a concrete formula for the density of instantons in terms of the Λ_{QCD} parameter, and a concrete form of $f(\vartheta)$. The function $f(\vartheta)$ in eq.(2.46) is given in an indirect way. One has to first solve the equation determining the phase of the saddle point of $N_\pm$, $\psi = \log(N_+/N_-)/2i$,

$$\psi \;+\; \frac{b-4}{4} \arctan \frac{(1-r^2)\sin\psi}{\cos\psi + r\sqrt{1 - r^2 \sin^2\psi}} \;=\; \vartheta, \qquad r \equiv \frac{\gamma_a}{\gamma_s}, \tag{2.48}$$

substitute its solution to determine the saddle point value of the partition function, and then read off $f(\vartheta)$,

$$f(\vartheta) \;=\; \cos\psi \left(\frac{r\cos\psi + \sqrt{1 - r^2\sin^2\psi}}{1 + r} \right)^{\frac{4-b}{b}}. \tag{2.49}$$

This function depends on the ratio, r, of the average interactions of same and opposite kind pseudoparticles, eqs.(2.34, 2.35).

It should be noted that not all values of $\gamma_{s,a}$ lead to a stable ensemble. First, $\gamma_s + \gamma_a$ must be positive — otherwise the system would collapse. This can be seen from eqs.(2.44, 2.47). Second, γ_s itself must be positive. If the interaction between same–kind pseudoparticles were on the average attractive, the system would prefer to break up into two subsystems, with only I's in one part of the universe and only $\bar{I}$'s in the other, with an apparent maximum violation of CP symmetry in both parts. Third, the topological susceptibility of the vacuum, eq.(2.23), must be positive. Determining the second derivative of $f(\vartheta)$ from eqs.(2.48, 2.49), we find

$$\langle Q_t^2 \rangle \;=\; \frac{4}{b - r(b-4)}\langle N \rangle_0. \tag{2.50}$$

Hence we must require $r < b/(b-4)$, meaning that the effective repulsion of opposite–kind pseudoparticles can not be much stronger than that of same–kind.

In several simple cases, the function $f(\vartheta)$ can be found analytically. For example, at $r = 1$, i.e., $\gamma_a = \gamma_s$, which is the case of the sum ansatz of [5], we have

$$f(\vartheta)_{r=1} \;=\; (\cos\vartheta)^{\frac{4}{b}}, \tag{2.51}$$

and the topological susceptibility is

$$\langle Q_t^2 \rangle \;=\; \langle N \rangle_0. \tag{2.52}$$

In the case $r = 0$ (no effective repulsion between opposite–kind instantons), we have

$$f(\vartheta)_{r=0} \;=\; \cos\frac{4\vartheta}{b}. \tag{2.53}$$

At $r \to -1$, which is the edge of the stability region, we get

$$f(\vartheta)_{r\to-1} \;=\; \left[\cos\frac{2\vartheta}{b-2}\right]^{\frac{2(b-2)}{b}}. \tag{2.54}$$

By measuring the ϑ–dependence of the partition function by lattice cooling, as described in section 2, one should be able to "experimentally" determine the ratio r. This quantity is one of the basic characteristics of the instanton vacuum.

To end this section, we present numerical values of the basic characteristics of the instanton vacuum, which were calculated using the two–loop instanton density, eq.(2.7). One first determines the average instanton sizes from eqs.(2.38, 2.44). The average instanton density then follows from maximizing eq.(2.41). Since the argument of the inverse

	a	$\left(\dfrac{N}{V\Lambda^4}\right)^{1/4}$	$\left(\overline{\rho^2}\dfrac{N}{V}\right)^{1/4}$	$\beta = b\log\left[a\left(\dfrac{N}{V\Lambda^4}\right)^{1/4}\right]$
I	2.5	.68	.37	5.8
	3	.66	.35	7.5
	3.5	.64	.34	9.0
	4	.63	.33	10.2
	5	.62	.31	12.4
	7	.60	.30	15.8
II	2.5	.74	.42	6.8
	3	.72	.40	8.5
	3.5	.71	.39	10.0
	4	.70	.38	11.3
	5	.68	.36	13.5
	7	.66	.34	16.9

Table 1: *The dimensionless instanton density, $(N/V\Lambda^4)^{1/4}$, the packing fraction, $(\overline{\rho^2} N/V)^{1/4}$, and the effective inverse charge, β, for various values of a parameter a, which defines the argument of the inverse charge according to $\beta = b\log\left[a(N/V\Lambda^4)^{1/4}\right]$. The two sets correspond to different values of the constant $\frac{1}{2}(\gamma_a + \gamma_s)$ determining the average instanton interaction, cf. eqs.(2.34, 2.35). Set I is obtained with the sum ansatz, eq.(2.36), set II with $\frac{1}{2}(\gamma_a + \gamma_s)$ equal to half the value of the sum ansatz.*

charge, β, is not uniquely determined from first principles, we set $\beta = b\log\left[a(N/V\Lambda^4)^{1/4}\right]$, with a a dimensionless parameter, and explore different choices of a. Results for the instanton density (*i.e.*, the gluon condensate), $(N/V\Lambda^4)^{1/4}$, the packing fraction, $(\overline{\rho^2} N/V)^{1/4}$, and the effective inverse charge are shown in table 1. Given there are the results for two different values of the constants defining the average instanton interactions, $\gamma_{a,s}$. Here, $N_+ = N_-$, so that the results depend only on the sum, $\gamma_a + \gamma_s$. From table 1 one sees that the instanton density and packing fraction depend only very weakly on the argument of the inverse coupling, a, although the values of β vary considerably. The values obtained for the density and packing fraction are rather close to those of [5].

3 Including fermions

3.1 The effective fermion action

So far we have discussed the partition function of instantons in pure gluodynamics. When fermions are included (N_f flavors), the coefficient of the beta function, eq.(2.4), is changed

to

$$b = \frac{11}{3}N_c - \frac{2}{3}N_f. \tag{3.1}$$

The fixed–$N_\pm$ partition function with fermions, normalized to the perturbative partition function, is given as

$$Z_{N_\pm}^{\text{fermions}} = \frac{1}{N_+!N_-!}\int \prod_I^{N_+ + N_-} d\xi_I \, d_0(\rho_I)\exp(-U_{\text{int}})\,\text{Det}(m, M_{\text{cut}}), \tag{3.2}$$

$$\text{Det}(m, M_{\text{cut}}) \equiv \frac{\det(i\nabla\!\!\!\!/\,(\xi) + im)}{\det(i\partial\!\!\!/ + im)}\,\frac{\det(i\partial\!\!\!/ + iM_{\text{cut}})}{\det(i\nabla\!\!\!\!/\,(\xi) + iM_{\text{cut}})}. \tag{3.3}$$

Here, $m = \text{diag}(m_1, \ldots, m_{N_f})$ is the bare quark mass matrix, M_{cut} the Pauli–Villars regulator mass. Furthermore, $\nabla\!\!\!\!/\,(\xi)$ denotes the fermion Dirac operator in the background of the $N_\pm$–instanton configuration,

$$\nabla\!\!\!\!/\,(\xi) = \partial\!\!\!/ - iA\!\!\!/\,(x;\xi),$$
$$A(x;\xi) = \sum_I^{N_+} A_+(x;\xi_I) + \sum_{\bar{I}}^{N_-} A_-(x,\xi_{\bar{I}}). \tag{3.4}$$

(Here, $A_\pm$ are the fields of individual I's and $\bar{I}'s$, in singular gauge [5].) The eigenvalues of the Dirac operator contributing to the determinant can can be divided into "low" and "high" frequencies by introducing a splitting mass parameter, M_1,

$$\text{Det}(m, M_{\text{cut}}) = \text{Det}(m, M_1)\,\text{Det}(M_1, M_{\text{cut}}). \tag{3.5}$$

The value of M_1 will be chosen of order $1/\bar{\rho}$. The essence of the instanton description of chiral symmetry breaking is to treat the two factors in eq.(3.5) in different ways [13, 14]. The high–frequency part can approximately be factorized in one–instanton contributions, and is therefore included in the one–instanton weight of eq.(3.2). The low–frequency part, which is dominated by the fermionic zero modes associated with the instantons, must be averaged in the background of all $N_\pm$ instantons simultaneously. This leads to the delocalization of the zero modes, which is the mechanism of chiral symmetry breaking.

When the fermion determinant is included in the variational description of the interacting instanton partition function, it modifies the effective instanton size distribution in the medium (in addition to the change in the β–function, eq.(3.1)). This modification is of the order N_f/N_c [14]. To lowest order in this parameter (quenched approximation), one gets back the unmodified size distribution of gluodynamics, eq.(2.39). In the following we use the gluodynamics size distribution in one–loop approximation to average the low–momentum fermion determinant. With the sum ansatz, eq.(2.36), the effective size distributions for I's and $\bar{I}$'s coincide, and eq.(2.39), with eq.(2.3), becomes

$$d_\pm(\rho) = \text{const} \times \rho^{b-5}\exp\left[-\frac{b-4}{2}\frac{\rho^2}{\overline{\rho^2}}\right], \qquad \overline{\rho^2} = \overline{\rho_+^2} = \overline{\rho_-^2}. \tag{3.6}$$

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Since $b = O(N_c)$, the normalized size distribution reduces to a δ–function in the limit $N_c \to \infty$,

$$\left(\int d\rho_\pm \, d_\pm(\rho)\right)^{-1} d(\rho) \;\to\; \delta(\rho - \bar\rho) \qquad (b \to \infty), \qquad \bar\rho \;=\; \sqrt{\bar{\rho^2}}. \tag{3.7}$$

For simplicity, we shall use the δ–function distribution in the following calculations[4], *i.e.*, we replace all instanton sizes by $\bar\rho$.

The variational approximation to the fixed–$N_\pm$ partition function with fermions can then be written as

$$Z_{N_\pm}^{\text{fermions}} \;=\; Z_{N_\pm} \, \overline{\text{Det}}_{N_\pm}, \qquad \overline{\text{Det}}_{N_\pm} \equiv \int \prod_I^{N_+ + N_-} \frac{d^4 z_I}{V} dU_I \, \text{Det}(m, M_1). \tag{3.8}$$

Here, $Z_{N_\pm}$ is the variational partition function of gluodynamics. We remark that corrections to this factorized form of the partition function can in principle be calculated using the variational principle.

In the ground state described by the partition function, eq.(3.8), the chiral symmetry of the fermions is spontaneously broken [13, 14]. A useful concept is the effective fermion action [17]. It is obtained by performing in eq.(3.8) the integral over the instanton coordinates, and is valid in the vicinity of the saddle point of eq.(3.8). In terms of the effective action, the fixed–$N_\pm$ partition function represented as

$$\overline{\text{Det}}_{N_\pm} \;=\; C \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \exp\left(-S_{\text{eff}}[\psi^\dagger, \psi]\right). \tag{3.9}$$

(Here, C is a normalization constant.) The effective action can be used to calculate averages of operators consisting of fermion fields (see section 4).

To explicitly construct the effective fermion action, we need to know the fermion propagator in the background of the $N_\pm$–instanton configuration, eq.(3.4). For a single instanton, the fermion propagator is singular in the chiral limit due to the zero mode. It may be approximated as the sum of the free propagator and the explicit contribution of the zero mode [14, 17],

$$\left(i\slashed{\nabla}\left(\xi_{I(\bar I)}\right) + im\right)^{-1}_{1-\text{inst}} \;\approx\; (i\slashed{\partial})^{-1} \;-\; \frac{\Phi_\pm(x; \xi_{I(\bar I)})\Phi_\pm^\dagger(y; \xi_{I(\bar I)})}{im}. \tag{3.10}$$

Here, $\Phi_\pm(x; \xi_{I(\bar I)})$ is the wave function of the fermion zero mode in the background of one $I(\bar I)$. This interpolating formula should be accurate both at small momenta $(p \ll 1/\bar\rho)$, where the zero mode is dominant, and at large momenta $(p \gg 1/\bar\rho)$, where the propagator reduces to the free one. Eq.(3.10) is equivalent to using an approximate fermion action [17],

$$\exp\left(-\widetilde{S}^{I(\bar I)}[\psi^\dagger, \psi]\right) \;\propto\; \exp\left(\sum_f^{N_f} \int d^4 x \, \psi_f^\dagger i\slashed{\partial}\, \psi_f\right) \prod_{f=1}^{N_f} \left(im_f - V_\pm^{I(\bar I)}[\psi_f^\dagger, \psi_f]\right), \tag{3.11}$$

---

[4]This approximation is only a practical simplification, and does not imply that we take b to infinity elsewhere. Corrections for finite–width size distributions can readily be included.

$$V_\pm^{I(\bar{I})}[\psi_f^\dagger, \psi_f] \;=\; \int d^4x \left(\psi_f^\dagger(x) i\!\!\not\!\partial\, \Phi_\pm(x; \xi_{I(\bar{I})}) \right) \int d^4y \left(\Phi_\pm^\dagger(y; \xi_{I(\bar{I})}) i\!\!\not\!\partial\, \psi_f(y) \right). \quad (3.12)$$

The fermion Green function calculated with this action coincides with eq.(3.10). Given the diluteness of the instanton medium, one may thus approximate the fermion action in the background of an $N_\pm$–instanton configuration by the product

$$\exp\left(-\widetilde{S}[\psi^\dagger, \psi]\right) \;\propto\; \exp\left(\sum_f^{N_f} \int d^4x\, \psi_f^\dagger i\!\!\not\!\partial\, \psi_f \right) \qquad\qquad (3.13)$$

$$\times \left(\prod_I^{N_+} \prod_{f=1}^{N_f} \left(im_f - V_\pm^I[\psi_f^\dagger, \psi_f] \right) \right) \left(\prod_{\bar{I}}^{N_-} \prod_{f=1}^{N_f} \left(im_f - V_\pm^{\bar{I}}[\psi_f^\dagger, \psi_f] \right) \right).$$

With the fermion propagator approximated by eq.(3.13), the average over the configurations of the independent instantons, eq.(3.8), reduces to a product of averages over individual I's and $\bar{I}$'s and can be carried out explicitly. Since the splitting mass, M_1, is of the order of the effective momentum cutoff in the zero mode wave function, $1/\bar{\rho}$, the normalizing determinant of eq.(3.3) in the zero mode approximation may simply be replaced by M_1 to the power of the number of zero modes,

$$\det(i\!\!\not\!\nabla + iM_1) \;\propto\; (iM_1)^{N_f(N_+ + N_-)}. \qquad\qquad (3.14)$$

Its only role in the following will be to make the partition function dimensionless. The averaged fermion determinant, eq.(3.8), becomes

$$\overline{\mathrm{Det}}_{N_\pm} \;=\; (iM_1)^{-N_f(N_+ + N_-)} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \int \prod_I^{N_+ + N_-} \frac{d^4 z_I}{V} dU_I \exp\left(-\widetilde{S}[\psi^\dagger, \psi]\right)$$

$$=\; (iM_1)^{-N_f(N_+ + N_-)} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp\left(\sum_f^{N_f} \int d^4x\, \psi_f^\dagger i\!\!\not\!\partial\, \psi_f \right) W_+^{N_+} W_-^{N_-}. \,(3.15)$$

Here, $W_\pm$ denote the one–instanton averages

$$W_\pm[\psi^\dagger, \psi] \;\equiv\; \int \frac{d^4 z_{I(\bar{I})}}{V} dU_{I(\bar{I})} \prod_f^{N_f} \left(im_f - V_\pm^{I(\bar{I})}[\psi_f^\dagger, \psi_f] \right). \qquad (3.16)$$

Performing the average over color to leading order in $1/N_c$, eq.(3.16) can be expressed as[5]

$$W_\pm \;=\; \left(-\frac{4\pi^2 \bar{\rho}^2}{N_c} \right)^{N_f} \int \frac{d^4 z}{V} \det J_\pm(z) \;+\; \sum_{f=1}^{N_f} im_f \left(-\frac{4\pi^2 \bar{\rho}^2}{N_c} \right)^{N_f - 1} \int \frac{d^4 z}{V} {\det}'_f J_\pm(z)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.17)$$

$$=\; i^{N_f} \left(\frac{4\pi^2 \bar{\rho}^2}{N_c} \right)^{N_f} \int \frac{d^4 z}{V} \det\left[iJ_\pm(z) + \frac{mN_c}{4\pi^2 \bar{\rho}^2} \right] \;+\; O(m^2). \qquad (3.18)$$

---

[5]Explicit expressions for the zero mode vertex, eq.(3.12) and formulas for integrals over the color orientation matrices can be found in [14, 17].

Here the determinants are over flavor indices, and $\det'_f$ denotes the minor in which the f–th row and column are omitted. The currents, $J_\pm(z)$, are color singlets and $N_f \times N_f$– matrices in flavor,

$$J_\pm(x)_{fg} = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \exp(-i(k-l)\cdot x)\, F(k)F(l)\, \psi_f^\dagger(k)\tfrac{1}{2}(1 \pm \gamma_5)\psi_g(l), \quad (3.19)$$

where $F(k)$ is a form factor related to the wave function of the zero mode in momentum space,

$$F(k) = -t\frac{d}{dt}\left[I_0(t)K_0(t) - I_1(t)K_1(t)\right] \rightarrow \begin{cases} 1 & t \to 0 \\ \frac{3}{4}t^{-3} & t \to \infty \end{cases} \quad (3.20)$$

$$t = \tfrac{1}{2}k\bar\rho.$$

The first term of eq.(3.17) has the form of the 't Hooft interaction [2]. The form eq.(3.18) is equal to eq.(3.17) to leading order in m.

To determine the effective fermion action, we have to find the saddle point of the fermion integral, eq.(3.15). We introduce an integral representation for the powers of the one–instanton averages,

$$\overline{\mathrm{Det}}_{N_\pm} = \int \frac{d\lambda_+}{2\pi} \int \frac{d\lambda_-}{2\pi} \exp\left(N_+ \log\left[\frac{N_+}{\lambda_+ V}\left(\frac{4\pi^2\bar\rho^2}{N_c M_1}\right)^{N_f}\right] - N_+ + (+\to -)\right) \quad (3.21)$$

$$\times \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp \int d^4x \left(\sum_f^{N_f} \psi_f^\dagger i\slashed{\partial}\psi_f + \lambda_+ \det\left[iJ_+(x) + \frac{mN_c}{4\pi^2\bar\rho^2}\right] + (+\to -)\right).$$

We now determine the common saddle point of eq.(3.21) in the variable $\lambda_\pm$ and the fermion field, in the thermodynamic limit, $N_\pm \to \infty, V \to \infty, N_\pm/V$ fixed.

The case $N_f = 1$. Let us first consider the case of one quark flavor, $N_f = 1$. In this case, the exponent in the fermion integral of eq.(3.21) is bilinear in the fermion field,

$$\int d^4z \left(iJ_\pm(z) + \frac{mN_c}{4\pi^2\bar\rho^2}\right) = i\int \frac{d^4k}{(2\pi)^4}\psi^\dagger(k)\tfrac{1}{2}(1\pm\gamma_5)\psi(k)F^2(k) + \frac{mVN_c}{4\pi^2\bar\rho^2}, (3.22)$$

and the fermion integral can be carried out exactly,

$$\overline{\mathrm{Det}}_{N_\pm} = \int \frac{d\lambda_+}{2\pi} \int \frac{d\lambda_-}{2\pi} \exp\left(-W_1(\lambda_+, \lambda_-)\right),$$

$$W_1(\lambda_+, \lambda_-) = -\left(N_+ \log\left[\frac{N_+}{\lambda_+ V}\left(\frac{4\pi^2\bar\rho^2}{N_c M_1}\right)\right] - N_+ + (+\to -)\right) - \frac{mVN_c}{4\pi^2\bar\rho^2}(\lambda_+ + \lambda_-)$$

$$- \mathrm{Tr}\log\left[\frac{\slashed{k} - \frac{1}{2}i\left[\lambda_+(1+\gamma_5) + \lambda_-(1-\gamma_5)\right]F^2(k)}{\slashed{k} - im}\right]. \quad (3.23)$$

(Here, Tr includes the trace over momentum space, and color and spinor indices.) Since the fluctuations of $\Delta = N_+ - N_-$ are small relative to $N = N_+ + N_-$ in the thermodynamic limit, it is sufficient to evaluate the effective action to leading order in Δ. We set

$$\lambda_\pm = M(1 \pm \delta), \quad (3.24)$$

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at the saddle point, with $\delta = O(\Delta)$. The values of M and δ are determined by minimizing the effective potential, eq.(3.23). This leads to the condition

$$4N_c V \int \frac{d^4 k}{(2\pi)^4} \frac{M^2 F^4(k)}{M^2 F^4(k) + k^2} = N - \frac{mMV N_c}{2\pi^2 \bar{\rho}^2}. \tag{3.25}$$

This equation is identical to the self–consistency condition of [14]. Note that M tends to a finite value in the thermodynamic limit. Parametrically,

$$M \propto \left(\frac{N}{V N_c}\right)^{1/2} \bar{\rho}. \tag{3.26}$$

Furthermore,

$$\delta = \frac{2\pi^2 \bar{\rho}^2 \Delta}{mMV N_c}, \tag{3.27}$$

at the saddle point. The effective fermion action, eq.(3.9), is thus determined as

$$S_{\text{eff}}[\psi^\dagger, \psi]_{N_f=1} = \int \frac{d^4 k}{(2\pi)^4} \psi^\dagger(k) \left[\not{k} - iMF^2(k)(1 + \delta\gamma_5)\right] \psi(k). \tag{3.28}$$

It describes fermions with a dynamical momentum–dependent mass, which is proportional to the square of the wave function of the instanton zero mode. For $\Delta \neq 0$, *i.e.*, for an unequal number of I's and $\bar{I}$'s, the fermion action shows a parity–violating mass term. This term diverges in the chiral limit; we have retained only the infrared–singular part and dropped all finite terms. This infrared singularity is canceled when averaging over Δ in the grand canonical ensemble and does not affect infrared–stable physical quantities (see section 4.2).

The case $N_f > 1$. In the case of more than one quark flavor, $N_f > 1$, the instanton–fermion vertex, eq.(3.18), describes a many–fermionic interaction ($2N_f$ fields). In order to perform the integral over the fermion field, we have to introduce auxiliary boson fields to linearize the exponent of eq.(3.21). This can be done in the large–N_c limit, using the formula

$$\exp(\lambda \det[iJ]) = \int d\mathcal{M} \exp\left[-(N_f - 1)\lambda^{-\frac{1}{N_f-1}} (\det \mathcal{M})^{\frac{1}{N_f-1}} + i\text{tr}(\mathcal{M}J)\right], \tag{3.29}$$

which holds in saddle point approximation. Here, $\mathcal{M}$ is a hermitean $N_f \times N_f$–matrix variable. With the help of eq.(3.29), eq.(3.21) is represented as

$$\overline{\text{Det}}_{N_\pm} = \int \frac{d\lambda_+}{2\pi} \int \frac{d\lambda_-}{2\pi} \exp\left(N_+ \log\left[\frac{N_+}{\lambda_+ V} \left(\frac{4\pi^2 \bar{\rho}^2}{N_c M_1}\right)^{N_f}\right] - N_+ + (+ \to -)\right) \tag{3.30}$$

$$\times \int \mathcal{DM}_+ \mathcal{DM}_- \exp \int d^4 x \left(-(N_f - 1)\lambda_+^{-\frac{1}{N_f-1}} (\det \mathcal{M}_+(x))^{\frac{1}{N_f-1}} - (+ \to -)\right.$$

$$\left. + \frac{N_c}{4\pi^2 \bar{\rho}^2} \text{tr}\left[m(\mathcal{M}_+(x) + \mathcal{M}_-(x))\right]\right)$$

$$\times \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp \int d^4 x \left(\sum_f^{N_f} \psi_f^\dagger i\not{\partial} \psi_f + i\text{tr}\left[\mathcal{M}_+(x)J_+(x) + \mathcal{M}_-(x)J_-(x)\right]\right).$$

One can now integrate over the fermion field. Furthermore, one may integrate over the auxiliary variables, $\lambda_\pm$, in saddle point approximation, and represent the partition function as

$$\overline{\mathrm{Det}}_{N_\pm} = \int \mathcal{D}\mathcal{M}_+ \mathcal{D}\mathcal{M}_- \exp\left(-W[\mathcal{M}_+, \mathcal{M}_-]\right). \tag{3.31}$$

For constant meson fields, $\mathcal{M}_\pm(x) \equiv \mathcal{M}_\pm$, the saddle point condition for $\lambda_\pm$ is

$$\lambda_\pm = \left(\frac{V}{N_\pm}\right)^{N_f - 1} \det \mathcal{M}_\pm. \tag{3.32}$$

Assuming the vacuum meson field to be diagonal in flavor, $\mathcal{M}_\pm = \mathrm{diag}(\mathcal{M}_{\pm 1}, \ldots \mathcal{M}_{\pm N_f})$, the effective potential is obtained as

$$
\begin{aligned}
W(\mathcal{M}_+, \mathcal{M}_-) = \ & -\sum_f^{N_f} \left(N_+ \log \left[\frac{N_+}{\mathcal{M}_{+f} V} \left(\frac{4\pi^2 \bar{\rho}^2}{N_c M_1} \right) \right] - N_+ \ + \ (+ \to -) \right) \tag{3.33} \\
& - \frac{N_c V}{4\pi^2 \bar{\rho}^2} \sum_f^{N_f} m_f (\mathcal{M}_{+f} + \mathcal{M}_{-f}) \\
& - \sum_f^{N_f} \mathrm{Tr} \log \left[\frac{\slashed{k} - \frac{1}{2} i \left[\mathcal{M}_{+f}(1 + \gamma_5) + \mathcal{M}_{-f}(1 - \gamma_5) \right] F^2(k)}{\slashed{k} - im} \right].
\end{aligned}
$$

To leading order in N_c, the fermion determinant is simply the product of the determinants for the different fermion flavors, and the effective potential is the sum of the potentials for one flavor, eq.(3.23). Setting

$$\mathcal{M}_{\pm f} = M_f(1 \pm \delta_f), \tag{3.34}$$

the minimization of eq.(3.34) thus leads to the same conditions as for $N_f = 1$, eqs.(3.25, 3.27), for each flavor. Substituting this vacuum value of the meson field in eq.(3.32), and returning to the representation eq.(3.21), we obtain the effective fermion action in the chiral limit,

$$S_{\mathrm{eff}}[\psi^\dagger, \psi] = -\left(\int d^4x \sum_f^{N_f} \psi_f^\dagger i \slashed{\partial} \psi_f \ + \ (1 + \delta) Y_+ \ + \ (1 - \delta) Y_- \right), \tag{3.35}$$

$$Y_\pm = \left(\frac{2V}{N} \right)^{N_f - 1} (iM)^{N_f} \int d^4x \, \det J_\pm(x), \tag{3.36}$$

$$\delta = \sum_f^{N_f} \delta_f = \frac{2\pi^2 \bar{\rho}^2 \Delta}{MV N_c} \left(\sum_f^{N_f} m_f^{-1} \right). \tag{3.37}$$

This general formula includes as a special case the result for $N_f = 1$, eq.(3.28). The dynamical fermion mass is the same as for $N_f = 1$. (Again, we have dropped $O(m)$ terms in the CP–symmetric part of the action, so that $M_1 = \ldots = M_{N_f} = M$.)

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3.2 Fermions and the topological susceptibility

In deriving the effective fermion action for given $N_{\pm}$, we have used the instanton size distribution of pure gluodynamics, eq.(3.6), an approximation justified by the formal parameter N_f/N_c. Indeed, the modifications of the size distribution due to the low–energy fermions are only quantitative. However, when considering the grand partition function, eq.(2.1), there is one important aspect in which fermions qualitatively alter the picture. The presence of dynamical light fermions modifies the probability of topological fluctuations, $\Delta = N_+ - N_- \neq 0$, in a principal way: $\langle \Delta^2 \rangle$ must vanish in the chiral limit. This remarkable property is a consequence of the $U(1)_A$–anomaly of QCD. In the following, we derive the topological susceptibility from the QCD anomaly equation and the chiral Ward identities. We then show, that the instanton vacuum leads to an identical result in the chiral limit.

The anomalous divergence of the isosinglet axial current operator is given by

$$\sum_f^{N_f} \partial^\mu (\bar{\psi}_f \gamma_\mu \gamma_5 \psi_f) \;\; = \;\; \frac{N_f}{16\pi^2} F\tilde{F} \; + \; 2i \sum_f^{N_f} m_f \bar{\psi}_f \gamma_5 \psi_f. \tag{3.38}$$

For the time–ordered product of two topological charge densities, one obtains the Ward identity [32]

$$-2i \sum_f^{N_f} m_f \int d^4x \, T\langle 0| F\tilde{F}(x), \, \bar{\psi}_f \gamma_5 \psi_f(0)|0\rangle \;\; = \;\; \frac{N_f}{16\pi^2} \int d^4x \, T\langle 0| F\tilde{F}(x), F\tilde{F}(0)|0\rangle. \tag{3.39}$$

To determine the topological susceptibility, we saturate the left–hand side of eq.(3.39) with pseudoscalar meson states. In general, there are N_f^2 pseudo–scalar bosons related to the breaking of $U(N_f) \times U(N_f)$, of which $N_f^2 - 1$ become massless in the chiral limit, (Goldstone bosons), while one singlet state remains massive. The couplings of these states to the non-singlet and singlet axial currents can be written as

$$q_\mu \, \langle 0| \bar{\psi} t^a \gamma_\mu \gamma_5 \psi(0) \, |\pi_A(q)\rangle \;\; \equiv \;\; F_\pi M_A^2 X_A^a, \qquad (a = 0, \ldots, N_f^2 - 1). \tag{3.40}$$

Here, the flavor generators, t^a, are normalized according to

$$\mathrm{tr} \left[t^a t^b \right] \;\; = \;\; \tfrac{1}{2} \delta^{ab}. \tag{3.41}$$

A unitary mixing matrix, X_A^a, takes into account that, for $m \neq 0$, mass and flavor eigenstates are in general not identical. (The index A runs over the N_f^2 "nonet" meson states.) We now need the matrix elements of the operators on the left–hand side of eq.(3.39) between the vacuum and one–meson states. The topological charge density, $F\tilde{F}$, is a flavor singlet, and its matrix element can be parametrized as

$$\langle 0|F\tilde{F}(0)|\pi_A\rangle \;\; \equiv \;\; \frac{\kappa F_\pi}{\sqrt{2N_f}} X_A^0, \tag{3.42}$$

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where κ is a constant, to be determined below. The matrix element of the pseudoscalar density, $\bar{\psi}t^a\gamma_5\psi$, is obtained from the non-singlet Ward identity,

$$-i\int d^4x\, T\langle 0|\,\bar{\psi}\,\{t^a,m\}_+\gamma_5\psi(x),\,\bar{\psi}t^b\gamma_5\psi(0)\,|0\rangle \quad = \quad \langle 0|\,\bar{\psi}\,\{t^a,t^b\}_+\psi(0)\,|0\rangle. \quad (3.43)$$

Inserting the pseudoscalar mass states, π_A, and using the definition of the mixing matrix, eq.(3.40), one finds

$$\langle 0|\bar{\psi}t^a\gamma_5\psi|\pi_A\rangle \quad = \quad i\frac{\langle\bar{\psi}\psi\rangle}{F_\pi}X_A^a. \quad (3.44)$$

Here,

$$\langle\bar{\psi}\psi\rangle \quad \equiv \quad \frac{1}{N_f}\sum_f^{N_f}\langle\bar{\psi}_f\psi_f\rangle. \quad (3.45)$$

Finally, the mixing matrix, X_A^a, is determined by taking the matrix element of the anomaly equation, eq.(3.38), and the non-anomalous divergence of the non-singlet axial current,

$$\partial^\mu(\bar{\psi}\gamma_\mu\gamma_5 t^a\psi) \quad = \quad i\bar{\psi}\{t^a,m\}_+\gamma_5\psi \qquad (a=1,\dots,N_f^2-1), \quad (3.46)$$

which leads to an equation

$$\sum_A M_A^2\, X_A^a\, X_A^{b*} \quad = \quad -\frac{2\langle\bar{\psi}\psi\rangle}{F_\pi^2}\mathrm{tr}\left[\{m,t^a\}_+ t^b\right] \;+\; \kappa\delta^{a0}\delta^{b0}. \quad (3.47)$$

Inserting now singlet and non-singlet states into eq.(3.39), using eqs.(3.42, 3.44), one obtains

$$\frac{1}{16\pi^2}\int d^4x\, T\langle 0|F\widetilde{F}(x),F\widetilde{F}(0)|0\rangle \quad = \quad \frac{\kappa F_\pi^2}{2N_f}\left(1-\kappa\sum_A\frac{X_A^0\,X_A^{0*}}{M_A^2}\right). \quad (3.48)$$

Solving eq.(3.47) for X_A^a, one obtains after a straightforward calculation

$$\frac{\langle Q_t^2\rangle}{V} \quad = \quad \frac{\kappa\langle\bar{\psi}\psi\rangle}{\dfrac{2N_f}{F_\pi^2}\langle\bar{\psi}\psi\rangle - \kappa\left(\sum_f^{N_f} m_f^{-1}\right)}. \quad (3.49)$$

In the chiral limit, $m\to 0$, this becomes

$$\frac{\langle Q_t^2\rangle}{V} \quad = \quad -\langle\bar{\psi}\psi\rangle\left(\sum_f^{N_f} m_f^{-1}\right)^{-1}. \quad (3.50)$$

This replaces the gluodynamics formula, eq.(2.23), in the presence of light fermions. The topological susceptibility is proportional to the harmonic average of the quark masses, which means that it vanishes if (at least) one of the fermion flavors becomes massless.

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It is interesting to consider eq.(3.49) also in the "quenched" limit, *i.e.*, to switch off the fermions before letting the quark masses go to zero. (Formally, this corresponds to the limit $N_f \to 0$, with $\sum m_f^{-1} = O(N_f)$ and $\langle \bar\psi\psi \rangle = O(1)$.) In this case, eq.(3.49) reduces to

$$\kappa = \frac{2N_f \langle Q_t^2 \rangle_{\text{quenched}}}{F_\pi^2 V}. \tag{3.51}$$

Taking now the limit $m \to 0$, the mixing between singlet and non–singlet states disappears, and one obtains from eq.(3.47)

$$\kappa = M_{\text{singlet}}^2. \tag{3.52}$$

Eqs.(3.51, 3.52) are the well-known Witten–Veneziano formula for the topological susceptibility in quenched QCD [32].

In the instanton language, the vanishing of the topological susceptibility for $m \to 0$ can be understood as being due to the "unbalanced" fermionic zero modes in the fermion determinant at $\Delta \neq 0$. To see this explicitly, we calculate the fermion determinant in zero mode approximation, eq.(3.8), as a function of Δ. Evaluating the integral representation, eqs.(3.31, 3.33), at the saddle point defined by eqs.(3.25, 3.34, 3.37), we obtain, to leading order in $1/m$,

$$\overline{\text{Det}}_{N_\pm} \propto \exp\left(-\frac{\pi^2 \bar\rho^2 \Delta^2}{N_c M V} \sum_f^{N_f} m_f^{-1} \right). \tag{3.53}$$

On the other hand, the quark condensate, calculated from eq.(3.33) at $\Delta = 0$, is

$$\langle \bar\psi\psi \rangle_{\text{Minkowski}} = -i\langle \psi^\dagger \psi \rangle_{\text{Euclidean}} = \frac{1}{N_f} \sum_f^{N_f} \left(-\frac{1}{V} \frac{\partial}{\partial m_f} \log \overline{\text{Det}}_{N_\pm} \right)_{\Delta=0} = -\frac{N_c M}{2\pi^2 \bar\rho^2}. \tag{3.54}$$

One thus sees that the Δ–distribution described by the fermion determinant of eq.(3.8) precisely realizes the theorem eq.(3.50),

$$\langle \Delta^2 \rangle = \frac{N_c M V}{2\pi^2 \bar\rho^2} \left(\sum_f^{N_f} m_f^{-1} \right)^{-1} = -V \langle \bar\psi\psi \rangle \left(\sum_f^{N_f} m_f^{-1} \right)^{-1}. \tag{3.55}$$

Note that the Δ–distribution corresponding to the full variational partition function, eq.(3.8), including the gluonic part,

$$P(\Delta) \propto Z_{N_\pm}^{\text{fermions}} = Z_{N_\pm} \overline{\text{Det}}_{N_\pm} \propto \exp\left(-\frac{\Delta^2}{2N} \right) \exp\left(\frac{\Delta^2}{2V \langle \bar\psi\psi \rangle} \sum_f^{N_f} m_f^{-1} \right), \tag{3.56}$$

is dominated by the one from the fermion determinant, since

$$\frac{N}{V} \gg -\langle \bar\psi\psi \rangle \left(\sum_f^{N_f} m_f^{-1} \right)^{-1}, \tag{3.57}$$

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in the chiral limit.

The variational partition function of independent instantons, eq.(3.8), including the fermion determinant, thus correctly describes the dispersion of topological vacuum fluctuations, which in QCD is governed by the $U(1)_A$ anomaly. Contrary to the case of the trace anomaly, which is realized already in the quenched approximation (see section 2.1), the inclusion of the fermion determinant is essential here. In section 4, we shall see that the independent instanton partition function with fermions, eq.(3.8), can be used also to evaluate matrix elements of the topological charge in hadronic states consistently with the $U(1)_A$ anomaly.

4 Evaluating gluon operators

4.1 Effective operators for gluons

To extract information about hadrons, one has to evaluate correlation functions of hadronic currents, together with some operators consisting of fermion and gauge fields. The prescription for calculating vacuum averages of functions of the gauge field and the fermion field is defined by the grand canonical instanton partition function with fermions, with the fixed–$N_\pm$ partition function given by the variational approximation, eq.(3.8). The grand canonical averaging is performed in two steps. First, for a fixed number of I's and $\bar{I}$'s, one integrates over the collective coordinates of the instantons and over the fermion field, taking into account the dynamically generated quark mass. In the final step, one then averages over the number of I's and $\bar{I}$'s according to their dispersion in the medium. In this section, we investigate fixed–$N_\pm$ (canonical) averages of operators, specifically of such involving gluon fields. The averaging over the number of I's and $\bar{I}$'s will be discussed in the following section.

Let us consider first averages of operators involving only fermion fields. For example, the baryon correlation function is defined as the vacuum average of two baryon currents, consisting of N_c fermion fields coupled to a color singlet,

$$J_N(x) = \frac{1}{N_c!} \varepsilon^{\alpha_1 \ldots \alpha_{N_c}} \Gamma_{s_1 f_1, \ldots s_{N_c} f_{N_c}} \psi^{\alpha_1}_{s_1 f_1}(x) \ldots \psi^{\alpha_{N_c}}_{s_{N_c} f_{N_c}}(x), \qquad (4.1)$$

(Γ is a spin–flavor matrix) [21]. For a fixed number of instantons, $N_\pm$, the normalized average over the instanton coordinates and the fermion fields can be represented as an integral with the effective fermion action, eq.(3.35),

$$\left\langle J_N(x_1) J_N^\dagger(x_2) \right\rangle_{\text{fixed}-N_\pm}$$

$$\equiv \overline{\text{Det}}_{N_\pm}^{-1} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; J_N(x_1) J_N^\dagger(x_2) \int \prod_I^{N_+ + N_-} \frac{dz_I}{V} dU_I \exp\left(-\widetilde{S}[\psi^\dagger, \psi]\right) \qquad (4.2)$$

$$= \frac{\int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; J_N(x_1) J_N^\dagger(x_2) \exp\left(-S_{\text{eff}}[\psi^\dagger, \psi]\right)}{\int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; \exp\left(-S_{\text{eff}}[\psi^\dagger, \psi]\right)} \equiv \langle J_N(x_1) J_N^\dagger(x_2) \rangle_{\text{eff}}. \qquad (4.3)$$

Here and in the following, we use the symbol $\langle \ldots \rangle_{\text{eff}}$ to denote the normalized average of a function of the fermion field within the effective fermion theory, as defined by eq.(4.3).

The fermion integral can be evaluated to the order in $1/N_c$ to which one knows the effective action.

A new situation arises when calculating averages of operators containing gluon fields. Consider the nucleon matrix element of a gluon operator, $\mathcal{F}[A]$, e.g., the local gauge–invariant operators $F_{\mu\nu}F_{\mu\nu}$ or $F_{\mu\nu}\widetilde{F}_{\mu\nu}$, or functions thereof. The nucleon matrix element is obtained from the correlation function of $\mathcal{F}[A]$ with nucleon currents, eq.(4.1),

$$\left\langle J_N(x_1)J_N^\dagger(x_2)\mathcal{F}[A]\right\rangle_{\text{fixed}-N_\pm} \tag{4.4}$$

$$= \overline{\text{Det}}_{N_\pm}^{-1} \int \mathcal{D}\psi^\dagger\mathcal{D}\psi \, J_N(x)J_N^\dagger(x_2) \int \prod_I^{N_++N_-} \frac{dz_I}{V}dU_I \, \mathcal{F}[A(\xi)] \, \exp\left(-\tilde{S}[\psi^\dagger,\psi]\right).$$

Here, the function $\mathcal{F}[A]$ is evaluated with the gauge field of the $N_\pm$–instanton configuration, eq.(3.4), and averaged over the instanton coordinates together with the fermion action in the instanton background, eq.(3.13). We want to represent eq.(4.4) as an average within the effective fermion theory of the form of eq.(4.3):

$$\left\langle J_N(x_1)J_N^\dagger(x_2)\mathcal{F}[A]\right\rangle_{\text{fixed}-N_\pm} \tag{4.5}$$

$$\equiv \frac{\int \mathcal{D}\psi^\dagger\mathcal{D}\psi \, J_N(x_1)J_N^\dagger(x_2) \text{ "}\mathcal{F}\text{"}[\psi^\dagger,\psi] \, \exp(-S_{\text{eff}}[\psi^\dagger,\psi])}{\int \mathcal{D}\psi^\dagger\mathcal{D}\psi \, \exp(-S_{\text{eff}}[\psi^\dagger,\psi])}$$

$$= \left\langle J_N(x_1)J_N^\dagger(x_2)\text{"}\mathcal{F}\text{"}[\psi^\dagger,\psi]\right\rangle_{\text{eff}}.$$

Here, "$\mathcal{F}$"$[\psi^\dagger,\psi]$ is a (generally non-local) fermion operator, which represents the gluon operator, $\mathcal{F}[A]$, in the effective fermion theory. We shall call "$\mathcal{F}$"$[\psi^\dagger,\psi]$ the effective operator to $\mathcal{F}[A]$ and denote it by the symbol of the gluon operator put in quotation marks. As with the effective fermion action, eq.(3.35), the definition eq.(4.5) is understood in the sense of the $1/N_c$–expansion, i.e., the effective operator is defined only in the vicinity of the saddle point of the effective fermion theory.

There are several advantages in performing the averaging in the order of eq.(4.5), integrating over instanton coordinates first. One obtains a simple and transparent representation of correlation functions with gluon operators. Furthermore, once the effective operator for a given gluon operator has been determined, it can be inserted in correlation functions of many different hadronic currents. The resulting fermionic correlation functions can be evaluated in the chiral soliton picture of the nucleon [21].

To construct the effective operator, we have to perform in eq.(4.4) the integral over the coordinates of the $N_\pm$ instantons, and cast the remaining fermion integral in the form of an integral with the effective fermion action, eq.(3.35). Since the effective operator is defined only in combination with the effective fermion action, it is important that eq.(4.4) is evaluated consistently with the integral defining the effective fermion action, eq.(3.9), using the same zero–mode approximation for the fermion propagator, eq.(3.13).

For an $N_\pm$–instanton configuration, eq.(3.4), the function of the gauge field, $\mathcal{F}[A]$, can be decomposed in one–instanton and multi–instanton contributions,

$$\mathcal{F}[A(\xi)] = \sum_I^{N_+}\mathcal{F}[A_+(\xi_I)] + \sum_{\bar{I}}^{N_-}\mathcal{F}[A_-(\xi_{\bar{I}})] + \text{2-inst.} + \text{3-inst.} + \dots \tag{4.6}$$

When averaged over configurations of independent instantons, the multi–instanton terms are proportional to additional powers of the packing fraction of the instanton medium, $\bar{\rho}^4 N/V$, relative to the one–instanton terms. As the instanton medium is dilute, these contributions are numerically small. We retain only the one–instanton contributions in eq.(4.4). It will be seen that this is consistent with the approximations made in deriving the effective fermion theory.

In the following, let us consider for simplicity eq.(4.4) without the baryon currents, i.e., the vacuum average of $\mathcal{F}[A]$. With the zero mode approximation for the fermion propagator, eq.(3.13), we obtain in analogy to eq.(3.15)

$$
\begin{aligned}
& \langle \mathcal{F}[A] \rangle_{\text{fixed}-N_{\pm}} \\
&= \overline{\text{Det}}_{N_\pm}^{-1} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \int \prod_I^{N_+ + N_-} \frac{d^4 z_I}{V} dU_I \left(\sum_I^{N_+ + N_-} \mathcal{F}[A_\pm(\xi_I)] \right) \exp\left(-\widetilde{S}[\psi^\dagger, \psi] \right) \quad (4.7) \\
&= \overline{\text{Det}}_{N_\pm}^{-1} \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \exp\left(\sum_f^{N_f} \int d^4 x \, \psi_f^\dagger i\!\!\not\partial \, \psi_f \right) \\
&\quad \times \left(N_+ \, W_{\mathcal{F}+} \, W_+^{N_+ - 1} \, W_-^{N_-} \; + \; N_- \, W_{\mathcal{F}-} \, W_+^{N_+} \, W_-^{N_- - 1} \right). \quad (4.8)
\end{aligned}
$$

Again we have used the fact that the instantons are independent. Here, $W_\pm$ are the free one–instanton averages, eq.(3.16), while $W_{\mathcal{F}\pm}$ denote the one–instanton averages with an insertion of the function $\mathcal{F}[A]$, evaluated with the gauge field of $I(\bar{I})$

$$
W_{\mathcal{F}\pm}[\psi^\dagger, \psi] \; \equiv \; (-1)^{N_f} \int \frac{d^4 z_{I(\bar{I})}}{V} dU_{I(\bar{I})} \, \mathcal{F}[A_\pm(\xi_{I(\bar{I})})] \prod_f^{N_f} V_\pm^{I(\bar{I})}[\psi_f^\dagger, \psi_f] \; + \; O(m). \, (4.9)
$$

Here, $\mathcal{F}[A]$ is averaged over the collective coordinates of one $I(\bar{I})$ together with the zero mode fermion vertex, eq.(3.12), which leads to a coupling of the gluon operator to the fermion fields. In the important case that $\mathcal{F}[A]$ is a color–singlet function of the gauge field, it is independent of the color orientation of the instanton. The average over orientations is thus the same as in the free instanton–fermion vertex, eq.(3.17), and leads to a fermion vertex similar to the 't Hooft interaction,

$$
W_{\mathcal{F}\pm} \; = \; i^{N_f} \left(\frac{4\pi^2 \bar{\rho}^2}{N_c M} \frac{N}{2V} \right)^{N_f} \frac{2}{N} Y_{\mathcal{F}\pm}, \quad (4.10)
$$

$$
Y_{\mathcal{F}\pm}[\psi^\dagger, \psi] \; = \; \left(\frac{2V}{N} \right)^{N_f - 1} (iM)^{N_f} \int d^4 z_{I(\bar{I})} \, \mathcal{F}[A_\pm(z_{I(\bar{I})}, U_{I(\bar{I})} = 1, \bar{\rho})] \, \det J_\pm(z). (4.11)
$$

(The prefactors have been chosen in analogy to the definition of the vertices, $Y_\pm$, eq.(3.36).) If $\mathcal{F}[A]$ is not a color singlet, which may be the case if it is part of a mixed fermion–gluon operator the average over color orientations in eq.(4.11) results in fermion vertices with non–singlet color structure.

We want to represent the fermion integral of eq.(4.8) as an integral with the effective fermion action, eq.(3.35). The effective action was derived from a product of N_+ I's and N_- $\bar{I}$'s. In eq.(4.8), however, only $N_\pm - 1$ of the $I(\bar{I})$'s are "free", while one $I(\bar{I})$ is

connected to the operator, $\mathcal{F}[A]$. The effective operator in the sense of eq.(4.5) can thus not be immediately identified with the vertices $Y_{\mathcal{F}\pm}$, eq.(4.11). Rather, it is of the form

$$\text{``}\mathcal{F}\text{''}[\psi^\dagger, \psi] \quad = \quad N_+ \, Y_{\mathcal{F}+} \, \mathcal{R}_+ \quad + \quad N_- \, Y_{\mathcal{F}-} \, \mathcal{R}_-, \tag{4.12}$$

where $\mathcal{R}_\pm[\psi^\dagger, \psi]$ are functions of the fermion field, which compensate the fact that we have "taken out" an $I(\bar{I})$ from the $N_\pm \, I(\bar{I})$'s that make the effective action, eq.(3.15).

To determine the factors $\mathcal{R}_\pm$, we proceed with the evaluation of eq.(4.8) in analogy to the derivation of the effective action, eq.(3.15). Using the same integral representation for the powers of one–instanton vertices, eq.(3.21), but this time for $N_\pm - 1$ "free" vertices, it is easy to see that $\mathcal{R}_\pm$ must be chosen as $Y_\mp$ times a factor which compensates the shift in the saddle point compared to the integral defining the effective fermion action, eq.(3.21). After a straightforward calculation one obtains

$$\mathcal{R}_\pm[\psi^\dagger, \psi] \quad = \quad \frac{4}{N^2} \, Y_\mp \, \exp\left[\left(1 - \frac{2}{N}(1+\delta)Y_+\right) + \left(1 - \frac{2}{N}(1-\delta)Y_-\right)\right] \tag{4.13}$$

$$= \quad \frac{Y_\mp}{\langle Y_+\rangle_{\mathrm{eff}} \langle Y_-\rangle_{\mathrm{eff}}} \, \exp\left[\left(1 - \frac{Y_+}{\langle Y_+\rangle_{\mathrm{eff}}}\right) + \left(1 - \frac{Y_-}{\langle Y_-\rangle_{\mathrm{eff}}}\right)\right]. \tag{4.14}$$

Here, $Y_\pm[\psi^\dagger, \psi]$ are the 't Hooft vertices of the effective fermion action, eq.(3.36). In the second equation we have used that, to leading order in $1/N_c$, the vacuum average of $Y_\pm$ is

$$\langle Y_\pm\rangle_{\mathrm{eff}} \quad = \quad \tfrac{1}{2} N(1 \mp \delta) \quad + \quad O(\Delta^2), \tag{4.15}$$

as follows from a straightforward calculation, using the saddle point equation, eq.(3.25).

The effective operator, as defined by eqs.(4.11, 4.12, 4.14), has an intuitively obvious form. It consists of 't Hooft–type vertices describing the interaction of the gluon operator with the fermions mediated by one instanton. The interaction strength is governed by M, the dynamical fermion mass. The one–instanton vertices are multiplied by certain universal functions of the fermion fields, $\mathcal{R}_\pm$. The meaning of these functions can be elucidated by expanding eq.(4.14) around the saddle point. To first order in $Y_\pm - \langle Y_\pm\rangle_{\mathrm{eff}}$, one has

$$\mathcal{R}_\pm \quad \simeq \quad \frac{1}{\langle Y_\pm\rangle_{\mathrm{eff}}} \left(1 - \frac{Y_\pm - \langle Y_\pm\rangle_{\mathrm{eff}}}{\langle Y_\pm\rangle_{\mathrm{eff}}}\right) \quad \simeq \quad Y_\pm^{-1}, \tag{4.16}$$

i.e., the functions $\mathcal{R}_\pm$ can be regarded as the "inverse" of a 't Hooft vertex in the vicinity of the saddle point. As will be seen below, the presence of the functions $\mathcal{R}_\pm[\psi^\dagger, \psi]$ in eq.(4.12), in addition to normalizing the effective operator, has important consequences when considering the connected average of the effective operator with other fermionic operators, such as hadronic currents.

Let us calculate the average of the effective operator, eq.(4.21), in the vacuum of the effective fermion theory. To leading order in $1/N_c$ only the disconnected average of the factors of eq.(4.12) needs to be considered. Furthermore, when averaging the exponential factor of eq.(4.14) we may use

$$\langle X^n\rangle_{\mathrm{eff}} \quad = \quad \langle X\rangle_{\mathrm{eff}}^n, \qquad \langle \exp X\rangle_{\mathrm{eff}} \quad = \quad \exp\langle X\rangle_{\mathrm{eff}} \tag{4.17}$$

(X is an arbitrary function of the fermion fields). With eq.(4.15), we obtain

$$\langle \mathcal{R}_\pm \rangle_{\text{eff}} \;=\; \langle Y_\pm \rangle_{\text{eff}}^{-1} \;=\; \frac{2}{N}(1 \pm \delta) \;+\; O(\Delta^2), \tag{4.18}$$

and thus

$$\langle \text{``}\mathcal{F}\text{''}[\psi^\dagger, \psi] \rangle_{\text{eff}} \;=\; N_+ \frac{\langle Y_{\mathcal{F}+} \rangle_{\text{eff}}}{\langle Y_+ \rangle_{\text{eff}}} \;+\; N_- \frac{\langle Y_{\mathcal{F}-} \rangle_{\text{eff}}}{\langle Y_- \rangle_{\text{eff}}}. \tag{4.19}$$

As expected, the average of the effective operator is given by the normalized average of the one–instanton vertices, multiplied by the number of I's and $\bar{I}$'s in the ensemble.

Let us construct the effective operators corresponding to the simplest gauge–invariant gluonic operators, $F_{\mu\nu}^2$ and $F_{\mu\nu}\widetilde{F}_{\mu\nu}$. For one $I(\bar{I})$,

$$F_{\mu\nu}^2(x) \;=\; \frac{192\rho^4}{[\rho^2 + (x-z)^2]^4}, \qquad F_{\mu\nu}\widetilde{F}_{\mu\nu}(x) \;=\; \pm F_{\mu\nu}^2(x), \tag{4.20}$$

and the corresponding one–instanton vertices, eq.(4.11), are

$$Y_{F^2,\pm}(x) \;=\; \left(\frac{2V}{N}\right)^{N_f - 1} (iM)^{N_f} \int d^4z \, \frac{192\rho^4}{[\rho^2 + (x-z)^2]^4} \, \det J_\pm(z), \tag{4.21}$$

$$Y_{F\widetilde{F},\pm}(x) \;=\; \pm Y_{F^2,\pm}(x). \tag{4.22}$$

In practice, we are interested only in matrix elements of these operators at zero momentum transfer (forward scattering), and it is sufficient to consider the integral of the local operators over the four–volume. The integrals of eqs.(4.21, 4.22) are identical to the 't Hooft vertices, eq.(3.36), times the action of one instanton,

$$\int d^4x \, Y_{F^2,\pm}(x) \;=\; 32\pi^2 \, Y_\pm, \qquad \int d^4x \, Y_{F\widetilde{F},\pm}(x) \;=\; \pm 32\pi^2 \, Y_\pm. \tag{4.23}$$

The vacuum averages of the effective operators, eq.(4.19), are thus

$$\left\langle \int d^4x \, \text{``}F_{\mu\nu}^2\text{''} \right\rangle_{\text{eff}} \;=\; 32\pi^2(N_+ + N_-) \;=\; \left\langle \int d^4x \, F_{\mu\nu}^2 \right\rangle_{\text{fixed}-N_\pm}, \tag{4.24}$$

and similarly for $F_{\mu\nu}\widetilde{F}_{\mu\nu}$, with $N_+ + N_-$ replaced by $N_+ - N_-$. The average of the effective operator in the vacuum of the effective fermion theory is thus identical to the average of the original operator in the fixed–$N_\pm$ instanton ensemble, in accordance with the definition, eq.(4.5).

Consider now vacuum averages of the effective operators with hadronic currents. An important property of the effective operator $\int d^4x \, \text{``}F_{\mu\nu}^2\text{''}$ is that its average together with an arbitrary fermionic operator, $\mathcal{O}[\psi^\dagger, \psi]$, reduces to a disconnected average,

$$\left\langle \mathcal{O} \int d^4x \, \text{``}F_{\mu\nu}^2\text{''} \right\rangle_{\text{eff}} \;=\; \langle \mathcal{O} \rangle_{\text{eff}} \left\langle \int d^4x \, \text{``}F_{\mu\nu}^2\text{''} \right\rangle_{\text{eff}} \tag{4.25}$$

$$=\; 32\pi^2(N_+ + N_-) \langle \mathcal{O} \rangle_{\text{eff}} \left[1 + O\left(\frac{1}{N_c}\right)\right].$$

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The connected part of the average, which would be of the same order in $1/N_c$ as the $1/N_c$–correction to the disconnected part in eq.(4.25), is zero. (Here, the terms "connected" and "disconnected" refer to the effective fermion theory.) To see this, let us calculate explicitly the connected part of eq.(4.25). The operator $\mathcal{O}$ can connect to any of the factors in the effective operator, eq.(4.21). The contribution of $\mathcal{O}$ connecting to the functions $\mathcal{R}_\pm$, eq.(4.14), is calculated using the rules

$$\langle \mathcal{O}, X^n \rangle_{\text{eff, conn}} = n\langle \mathcal{O}, X \rangle_{\text{eff, conn}} \langle X \rangle_{\text{eff}}^{n-1}, \tag{4.26}$$

$$\langle \mathcal{O}, \exp X \rangle_{\text{eff, conn}} = \langle \mathcal{O}, X \rangle_{\text{eff, conn}} \left(\frac{d}{dX} \exp X \right)_{X = \langle X \rangle_{\text{eff}}} = \langle \mathcal{O}, X \rangle_{\text{eff, conn}} \exp\langle X \rangle_{\text{eff}},$$

which apply to the leading order in $1/N_c$. One easily obtains

$$\langle \mathcal{O}, Y_\pm \mathcal{R}_\pm \rangle_{\text{eff, conn}} = \langle \mathcal{O}, Y_\pm \rangle_{\text{eff, conn}} \langle \mathcal{R}_\pm \rangle_{\text{eff}} + \langle Y_\pm \rangle_{\text{eff}} \langle \mathcal{O}, R_\pm \rangle_{\text{eff, conn}} = 0, \tag{4.27}$$

in accordance with the interpretation of $\mathcal{R}_\pm$ as the "inverse" of $Y_\pm$, eq.(4.16). From this it follows that

$$\langle \mathcal{O} \int d^4x \text{ "}F_{\mu\nu}^2\text{"} \rangle_{\text{eff, conn}} = 0. \tag{4.28}$$

An analogous property holds for connected averages with the operator $\int d^4x \text{ "}F_{\mu\nu}\widetilde{F}_{\mu\nu}\text{"}$.

The role of the effective operators $\int d^4x \text{ "}F_{\mu\nu}^2\text{"}$ and $\int d^4x \text{ "}F_{\mu\nu}\widetilde{F}_{\mu\nu}\text{"}$, when averaged together with other fermionic operators, is simply to multiply the average of these operators by $32\pi^2(N_+ \pm N_-)$. There are no other dynamical effects from the insertion of an operator $\int d^4x \text{ "}F_{\mu\nu}^2\text{"}$ or $\int d^4x \text{ "}F_{\mu\nu}\widetilde{F}_{\mu\nu}\text{"}$ in a correlation function in the effective fermion theory. This is what one should expect: the original operators $\int d^4x\, F_{\mu\nu}^2$ and $\int d^4x\, F_{\mu\nu}\widetilde{F}_{\mu\nu}$ measure just the number of I's and $\bar{I}$'s in the ensemble, and, by definition, the average of the effective operators in the effective theory is equal to the average of the original operator in the instanton ensemble with fermions.

The absence of a connected average in eq.(4.25) has important consequences when passing from canonical to grand canonical averages. This property is crucial for realizing the trace and $U(1)_A$–anomaly in the matrix elements of the operators $\int d^4x\, F_{\mu\nu}^2$ and $\int d^4x\, F_{\mu\nu}\widetilde{F}_{\mu\nu}$ in hadronic states, as will be seen in section 5.

4.2   Fluctuations of the numbers of instantons

The effective fermion action, together with the effective operators for gluons introduced in the previous section, allows to calculate normalized averages of operators in the ensemble with a fixed number of instantons. Physical quantities, however, are given by averages over the grand canonical averages, described by the grand partition function with fermions, eq.(2.1). To pass from canonical to grand canonical averages, we have to sum the fixed–$N_\pm$ normalized average, eqs.(4.2, 4.4), over the distribution of the numbers of I's and $\bar{I}$'s in the ensemble. The weight for a configuration with given N_+, N_- is given by the value of the fixed–$N_\pm$ partition function with fermions. With the variational approximation, eq.(3.8),

$$P(N_+, N_-) \propto Z_{N_\pm}^{\text{fermions}} = Z_{N_\pm} \overline{\text{Det}}_{N_\pm}. \tag{4.29}$$

(For convenience, we shall assume the distribution $P(N_+, N_-)$ to be normalized.) Since we consider only quadratic (gaussian) fluctuations around the equilibrium values, $N_+ = N_- = \frac{1}{2}\langle N \rangle$, we may instead of $N_\pm$ equivalently average over independent distributions of $N = N_+ + N_-$ and $\Delta = N_+ - N_-$.

The normalized grand canonical average of a fermionic operator, $\mathcal{O}[\psi^\dagger, \psi]$, is defined as

$$\left\langle \mathcal{O}[\psi^\dagger, \psi] \right\rangle \equiv \sum_{N_+, N_-} P(N_+, N_-) \left\langle \mathcal{O}[\psi^\dagger, \psi] \right\rangle_{\text{eff}}. \tag{4.30}$$

For a gluon operator, $\mathcal{F}[A]$, the only difference is that the canonical average is calculated with the help of the corresponding effective operator,

$$\langle \mathcal{F}[A] \rangle \equiv \sum_{N_+, N_-} P(N_+, N_-) \left\langle \text{``}\mathcal{F}\text{''}[\psi^\dagger, \psi] \right\rangle_{\text{eff}}. \tag{4.31}$$

Here, it is assumed that the fixed–$N_\pm$ averages over the effective fermion theory have been evaluated as functions of $N_\pm$, in the vicinity of $N_+ = N_- = \frac{1}{2}\langle N \rangle$ and to first order in $\Delta = N_+ - N_-$.

In particular, with this definition, the grand canonical average of $\int d^4x \, F_{\mu\nu}^2$, using eq.(4.24), is

$$\left\langle \frac{1}{32\pi^2} \int d^4x \, F_{\mu\nu}^2 \right\rangle = \sum_{N_+, N_-} P(N_+, N_-) \left\langle \frac{1}{32\pi^2} \int d^4x \, \text{``}F_{\mu\nu}^2\text{''}[\psi^\dagger, \psi] \right\rangle_{\text{eff}}$$

$$= \sum_{N_+, N_-} P(N_+, N_-) \, (N_+ + N_-) = \langle N \rangle. \tag{4.32}$$

The effective operator technique reproduces the usual value of the vacuum gluon condensate. In the same way one obtains that the grand canonical average of $\int d^4x \, F_{\mu\nu}\widetilde{F}_{\mu\nu}$ is zero.

The relative width of fluctuations of $N - \langle N \rangle$ and Δ is inversely proportional to the size of the system, *cf.* eqs.(2.20, 3.56). Thus, fluctuations have no effect on averages of intensive quantities in the thermodynamic limit. For example, the correlation function of two baryon currents in the grand canonical ensemble simply reduces to the one in the canonical ensemble, eq.(4.2), evaluated at the equilibrium value of $N_\pm$,

$$\left\langle J_N(x_1) J_N^\dagger(x_2) \right\rangle = \left\langle J_N(x_1) J_N^\dagger(x_2) \right\rangle_{\text{eff}} \Big|_{N=\langle N \rangle, \, \Delta=0} + O\left(\frac{1}{\langle N \rangle}, \frac{1}{V}\right). \tag{4.33}$$

As a consequence, the infrared–singular term of the effective fermion action proportional to Δ, eq.(3.35), does not affect free hadron correlation correlation functions, as it should be. However, fluctuations of N and Δ lead to non-trivial consequences, when considering connected grand canonical averages of fermionic currents with extensive operators, such as $\int d^4x \, F_{\mu\nu}^2$ and $\int d^4x \, F_{\mu\nu}\widetilde{F}_{\mu\nu}$. We shall now discuss these effects, considering separately non-topological ($N - \langle N \rangle$) and topological (Δ) fluctuations.

Non-topological fluctuations. Let us consider the connected grand canonical average of an arbitrary fermionic operator, $\mathcal{O}[\psi^\dagger, \psi]$, with the operator $\int d^4x \, F_{\mu\nu}^2$. According to

eqs.(4.30, 4.31),

$$\left\langle \mathcal{O} \frac{1}{32\pi^2} \int d^4x\, F_{\mu\nu}^2 \right\rangle - \langle \mathcal{O} \rangle \left\langle \frac{1}{32\pi^2} \int d^4x\, F_{\mu\nu}^2 \right\rangle$$

$$= \sum_N P(N) \left\langle \mathcal{O} \frac{1}{32\pi^2} \int d^4x\, "F_{\mu\nu}^2" \right\rangle_{\text{eff}}$$

$$- \left(\sum_N P(N) \langle \mathcal{O} \rangle_{\text{eff}} \right) \left(\sum_N P(N) \left\langle \frac{1}{32\pi^2} \int d^4x\, "F_{\mu\nu}^2" \right\rangle_{\text{eff}} \right)$$

$$= \sum_N P(N)(N - \langle N \rangle) \langle \mathcal{O} \rangle_{\text{eff}} \quad = \quad \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} \left(N \frac{d}{dN} \langle \mathcal{O} \rangle_{\text{eff}} \right)_{N=\langle N \rangle} . \quad (4.34)$$

(It is understood that $\Delta = 0$; we do not write the averaging over Δ here.) Here, we have employed the property eq.(4.25) for the fixed–N average. In the last equation, we have used the fact that the relative width of the fluctuations, $\langle (N - \langle N \rangle)^2 \rangle / \langle N \rangle$, becomes small in the thermodynamic limit. (In addition, the width is proportional to $1/N_c$, eq.(2.20)). In eq.(4.34) it is implied that $\langle \mathcal{O} \rangle_{\text{eff}}$ is first evaluated by integrating over the fermion fields for fixed N, and then differentiated with respect to N. This differentiation also includes the dependence of $\langle \mathcal{O} \rangle_{\text{eff}}$ on N through the instanton size, $\bar{\rho}$, which is related to N/V according to eq.(2.44). Since the only scale in the effective fermion theory is the instanton density, N/V, the operator $N(d/dN)$ measures the total scale dependence of $\langle \mathcal{O} \rangle_{\text{eff}}$. In this sense, it plays the role of $\Lambda(d/d\Lambda)$ in QCD.

Taking the N–distribution from gluodynamics (quenched approximation), eq.(2.20), eq.(4.34) becomes

$$\left\langle \mathcal{O} \frac{1}{32\pi^2} \int d^4x\, F_{\mu\nu}^2 \right\rangle - \langle \mathcal{O} \rangle \left\langle \frac{1}{32\pi^2} \int d^4x F_{\mu\nu}^2 \right\rangle \quad = \quad \frac{4}{b} \left(N \frac{d}{dN} \langle \mathcal{O} \rangle_{\text{eff}} \right)_{N=\langle N \rangle} . \quad (4.35)$$

We have reduced the connected grand canonical correlation function to the N–derivative of a fixed–N correlation function, calculable in the effective fermion theory at equilibrium, $N = \langle N \rangle$. Eq.(4.35) is the statement of the fact that the renormalization (scale dependence) properties of QCD are realized in our approach. As a special case, eq.(4.35) contains the low–energy theorem of scale invariance for the vacuum averages of local fermionic operators [33, 5]. If $\mathcal{O}$ is a local operator, then, as N/V is the only scale in the effective fermion theory,

$$\langle \mathcal{O} \rangle_{\text{eff}} \;\propto\; \left(\frac{N}{V} \right)^{d/4}, \qquad \left(N \frac{d}{dN} \langle \mathcal{O} \rangle_{\text{eff}} \right)_{N=\langle N \rangle} = \frac{d}{4} \langle \mathcal{O} \rangle_{\text{eff}} . \quad (4.36)$$

Here, d is the naive dimension of the operator.

We emphasize that eq.(4.35) is valid also for non–local fermionic operators, in particular, for operators of the form $\mathcal{O} = J_N(x_1) J_N^\dagger(x_2)$. Such operator have an intrinsic scale, defined by the distance of the two points, x_1, x_2, and the vacuum average is no longer a homogeneous function of N/V. However, eq.(4.35) still leads to useful results in the limit of large distance, $|x_1 - x_2| \to \infty$. In this regime, the correlation function possesses an asymptotic expansion, the coefficients of which are dimensionful and therefore proportional to powers of the instanton density. This fact can be used to derive the nucleon matrix element of $\int d^4x\, F_{\mu\nu}^2$ (see section 5).

To summarize, a correct realization of the scale dependence properties of QCD is achieved in our approach already in quenched approximation, neglecting the influence of the fermion determinant on the distribution of N. This is different for topological fluctuations, which we shall now discuss.

Topological fluctuations. We now consider correlation functions with the topological charge operator, $\int d^4x\, F_{\mu\nu}\tilde{F}_{\mu\nu}$. In this case, the average over fluctuations of $\Delta = N_+ - N_-$ leads to non-trivial consequences. In analogy to eq.(4.35), the grand canonical average of an arbitrary fermionic operator, $\mathcal{O}[\psi^\dagger,\psi]$, with the topological charge becomes

$$
\left\langle \mathcal{O}\frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}_{\mu\nu}\right\rangle = \sum_\Delta P(\Delta)\left\langle \mathcal{O}\frac{1}{32\pi^2}\int d^4x\, \text{``}F_{\mu\nu}\tilde{F}_{\mu\nu}\text{''}\right\rangle_{\text{eff}}
$$

$$
= \sum_\Delta P(\Delta)\,\Delta\,\langle\mathcal{O}\rangle_{\text{eff}} = \langle\Delta^2\rangle\left(\frac{d}{d\Delta}\langle\mathcal{O}\rangle_{\text{eff}}\right)_{\Delta=0}. \quad (4.37)
$$

Here, we have used that $\langle\Delta^2\rangle^{1/2}/N$ is small for large N. The derivative with respect to Δ of the fixed–$N_\pm$ average in the effective theory can be calculated as

$$
\left(\frac{d}{d\Delta}\langle\mathcal{O}\rangle_{\text{eff}}\right)_{\Delta=0} = \left\langle \mathcal{O}\frac{dS_{\text{eff}}}{d\Delta}\right\rangle_{\text{eff}}\bigg|_{\Delta=0}, \quad (4.38)
$$

$$
\frac{dS_{\text{eff}}}{d\Delta}[\psi^\dagger,\psi] = \frac{2\pi^2\bar{\rho}^2}{N_c V M}\left(\sum_f^{N_f} m_f^{-1}\right)(Y_+ - Y_-). \quad (4.39)
$$

Here, $dS_{\text{eff}}/d\Delta[\psi^\dagger,\psi]$ is treated as an operator insertion, and the average over the effective fermion is now to be taken with $\Delta = 0$.

For the dispersion of Δ in eq.(4.37), we must now use the distribution which includes the effect of the fermion determinant, eq.(3.55). With this dispersion, one obtains

$$
\left\langle \mathcal{O}\frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}_{\mu\nu}\right\rangle = \langle\mathcal{O}\,(Y_+ - Y_-)\rangle_{\text{eff}}\,|_{\Delta=0}. \quad (4.40)
$$

Notice that the $1/m$–divergence in the chiral limit, which is present in the effective fermion action for $\Delta \neq 0$, is compensated by the fact that the dispersion of Δ, eq.(3.55), is of order m. As a result, the right–hand side of eq.(4.40) is finite in the chiral limit. It would be senseless to average over Δ using the quenched distribution, $\langle\Delta^2\rangle = \langle N\rangle$ — the result would be divergent in the chiral limit. This is an important difference to the description of non-topological fluctuations.

We now want to demonstrate that eq.(4.40) is consistent with the $U(1)_A$ anomaly of QCD. To this end, let us derive the $U(1)$–axial current and its divergence in the effective fermion theory defined by eq.(3.35), at $\Delta = 0$. Consider a $U(1)$–chiral rotation with parameter ε,

$$
\psi \rightarrow \exp\left(i\varepsilon\gamma_5\right)\psi, \qquad \psi^\dagger \rightarrow \psi^\dagger\exp\left(i\varepsilon\gamma_5\right). \quad (4.41)
$$

The change of the effective action, eq.(3.35), gives the divergence of the axial current,

$$
\int d^4x\, \partial_\mu J_{5\mu}(x) = \frac{dS_{\text{eff}}}{d\varepsilon}\bigg|_{\varepsilon=0} = 2N_f(Y_+ - Y_-). \quad (4.42)
$$

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The current itself is obtained from the fermion kinetic term, as usual[6],

$$J_{5\mu}(x) = \sum_f^{N_f} \psi_f^\dagger(x)\gamma_\mu\gamma_5\psi_f(x). \tag{4.43}$$

Thus, eq.(4.40) can equivalently be written as

$$\left\langle \mathcal{O}\,\frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}\widetilde{F}_{\mu\nu}\right\rangle = \left\langle \mathcal{O}\,\frac{1}{2N_f}\int d^4x\,\partial_\mu J_{5\mu}(x)\right\rangle_{\text{eff}}\Big|_{\Delta=0}. \tag{4.44}$$

This is precisely the relation which in QCD would follow from the anomaly equation, eq.(3.38), in the chiral limit. We have thus shown that our prescription, based on the effective fermion theory and the effective operator for $\int d^4x\, F_{\mu\nu}\widetilde{F}_{\mu\nu}$, fully realizes the $U(1)_A$ anomaly at the level of correlation functions.

One should note the different character of the $U(1)_A$ symmetry breaking in QCD and the effective fermion theory. In QCD, the symmetry is broken anomalously, *i.e.*, by the high–momentum components of the fermion field. The effective fermion theory describes only the low–momentum components of the fermion field, up to a cutoff, $\bar{\rho}^{-1}$, and so, by definition, it has no anomaly. (We have assumed this in deriving the divergence of the axial current, eq.(4.42).) However, in the effective theory the $U(1)_A$ symmetry is broken explicitly, which results in a non-zero divergence of the axial current.

5 Hadronic matrix elements of $F_{\mu\nu}^2$ and $F_{\mu\nu}\widetilde{F}_{\mu\nu}$

We now apply the techniques developed in the previous sections to the study of the nucleon matrix elements of the operators $F_{\mu\nu}^2$ and $F_{\mu\nu}\widetilde{F}_{\mu\nu}$. On general grounds, the nucleon matrix elements of these operators can be parametrized as

$$\langle p'|F_{\mu\nu}^2(0)|p\rangle = A_S(q^2)\,m_N\,\bar{u}'u, \tag{5.1}$$

$$\langle p'|F_{\mu\nu}\widetilde{F}_{\mu\nu}(0)|p\rangle = A_P(q^2)\,m_N\,\bar{u}'i\gamma_5 u, \tag{5.2}$$

where $q = p' - p$, and u, u' denote nucleon spinors. In QCD, $A_S(0)$ is uniquely determined by the conformal anomaly [34],

$$A_S(0) = -\frac{32\pi^2}{b}, \tag{5.3}$$

whereas $A_P(0)$ is related to the isosinglet axial coupling constant, $g_A^{(0)}$, by virtue of the axial anomaly equation, eq.(3.38),

$$A_P(0) = \frac{32\pi^2}{N_f}g_A^{(0)}, \tag{5.4}$$

$$\langle p|J_{5\mu}|p\rangle = g_A^{(0)}\bar{u}\gamma_\mu\gamma_5 u. \tag{5.5}$$

---

[6]We neglect the contributions to the current due to the momentum dependence of the dynamical fermion mass. They are parametrically small, of order $M\bar{\rho}$.

We show now that these relations are realized in our approximate description. Thus, the renormalization properties of QCD and the axial anomaly are correctly taken into account even at the level of hadronic matrix elements.

Consider first the operator $F_{\mu\nu}^2(x)$. The nucleon matrix element in our approach is extracted from the large–time limit of the connected euclidean correlation function,

$$\left\langle J_N(T) J_N^\dagger(-T) F_{\mu\nu}^2(x) \right\rangle - \left\langle J_N(T) J_N^\dagger(-T) \right\rangle \left\langle F_{\mu\nu}^2(x) \right\rangle. \tag{5.6}$$

(Here, "connected" refers to the full grand canonical average.) For brevity, $J_N(T) \equiv J_N(\mathbf{0}, T)$, $etc.$ Since we are interested in the zero momentum transfer matrix element, we integrate the operator $F_{\mu\nu}^2(x)$ over the euclidean four–volume. One obtains

$$A_S(0) = \lim_{T\to\infty} \frac{1}{2m_N T} \frac{\left\langle J_N(T) J_N^\dagger(-T) \int d^4x\, F_{\mu\nu}^2 \right\rangle - \left\langle J_N(T) J_N^\dagger(-T) \right\rangle \left\langle \int d^4x\, F_{\mu\nu}^2 \right\rangle}{\left\langle J_N(T) J_N^\dagger(-T) \right\rangle}. \tag{5.7}$$

By virtue of the effective version of the "low–energy theorem", eq.(4.35), this becomes

$$A_S(0) = \lim_{T\to\infty} \frac{1}{2m_N T} 32\pi^2 \frac{4}{b} \left(N \frac{d}{dN} \log \left\langle J_N(T) J_N^\dagger(-T) \right\rangle_{\text{eff}} \right)_{N=\langle N\rangle}. \tag{5.8}$$

For large T, the euclidean correlation function decays exponentially,

$$\left\langle J_N(T) J_N^\dagger(-T) \right\rangle_{\text{eff}} \sim \exp\left(-2T m_N\right). \tag{5.9}$$

Here, m_N is the nucleon mass, as calculated in the effective fermion theory for a fixed number of instantons, N. For dimensional reasons,

$$m_N \propto \left(\frac{N}{V}\right)^{1/4}, \tag{5.10}$$

since the instanton density is the only scale in the effective fermion theory. (The instanton size, $\bar{\rho}$, is related to N/V by eq.(2.44).) With eq.(5.9) and eq.(5.10), we obtain from eq.(5.8)

$$A_S(0) = -\frac{32\pi^2}{b}, \tag{5.11}$$

in agreement with the trace anomaly. This result is again a manifestation of the fact that the instanton vacuum preserves the renormalization properties of QCD.

Let us now consider the nucleon matrix element of the topological charge, eq.(5.2). In analogy to eq.(5.7), it is obtained as

$$A_P(0) = \lim_{T\to\infty} \frac{1}{2m_N T} \frac{\left\langle J_N(T) J_N^\dagger(-T)\, F_{\mu\nu} \tilde{F}_{\mu\nu}(x) \right\rangle}{\left\langle J_N(T) J_N^\dagger(-T) \right\rangle}. \tag{5.12}$$

With the realization of the axial anomaly in the effective theory, eq.(4.44), we have

$$A_P(0) = \lim_{T \to \infty} \frac{1}{2m_N T} \frac{\frac{16\pi^2}{N_f} \left\langle J_N(T) J_N^\dagger(-T) \int d^4x\, \partial_\mu J_{5\mu}(x) \right\rangle_{\mathrm{eff}} \Big|_{\Delta=0}}{\left\langle J_N(T) J_N^\dagger(-T) \right\rangle_{\mathrm{eff}} \Big|_{\Delta=0}}. \qquad (5.13)$$

The correlation functions here are to be evaluated in the large–N_c limit, in which the nucleon is described as a chiral soliton in the effective fermion theory [21]. In momentum representation, the free nucleon correlation function develops a pole at m_N, and a nucleon state emerges in the effective theory. Saturating the correlation functions numerator and denominator of eq.(5.13) with one–nucleon states of the effective theory, one finally obtains

$$A_P(0) = \frac{32\pi^2}{N_f} g_A^{(0)}, \qquad (5.14)$$

where $g_A^{(0)}$ now denotes the axial coupling of the nucleon, as evaluated in the effective fermion theory from the matrix element of the isosinglet axial current, *cf.* eq.(5.5). We have thus shown, that the axial anomaly is realized in our effective description even at the level of hadronic matrix elements: the nucleon matrix element of $F_{\mu\nu}\widetilde{F}_{\mu\nu}$ reduces to the nucleon isosinglet axial coupling within our effective description.

The nucleon isosinglet axial coupling, $g_A^{(0)}$, has been evaluated numerically in the chiral quark soliton model of the nucleon, which is obtained from the above effective fermion theory by bosonization [21]. Calculations give $g_A^{(0)} \sim 0.36$ [27], which agrees well with the value obtained in a recent analysis of polarized nucleon structure functions[7] [35].

It has been suggested by Anselm [36] that a small value of $g_A^{(0)}$ could be explained by equating $A_P(0)$ with $A_S(0)$, and thus $g_A^{(0)}$ with $-N_f/b$, at the constituent quark level. He argued that $A_P(0) = A_S(0)$ could be obtained from the fact that $F_{\mu\nu}\widetilde{F}_{\mu\nu} = \pm F_{\mu\nu}^2$ for one $I(\bar{I})$. Our derivation shows that there is no justification for such a picture. The fluctuations of the numbers of instantons in the ensemble are crucial to consistently describe the matrix elements of $F_{\mu\nu}^2$ and $F_{\mu\nu}\widetilde{F}_{\mu\nu}$. For a single constituent quark instead of the nucleon, we immediately obtain from eq.(5.13) $g_A^{(0)} = 1$. The fact that a value of $g_A^{(0)}$ considerably smaller than 1 is obtained for the nucleon [27] can be understood as an effect of the meson cloud of the nucleon (polarization of the Dirac sea by the N_c valence quarks).

6 Conclusions and outlook

In this paper we have considered a description of non–perturbative phenomena in QCD in the form of a grand canonical ensemble of instantons. The fluctuations of the number of pseudoparticles play a crucial role in realizing the renormalization properties of QCD in the resulting statistical mechanics system. Fluctuations of $N = N_+ + N_-$ (non-topological) are related to the trace anomaly, while fluctuations of $\Delta = N_+ - N_-$ (topological) realize the $U(1)_A$–anomaly. The relation between the QCD anomalies and the fluctuations of $N_\pm$

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[7]For a review of the "proton spin problem", see [24].

has two directions: First, starting from the QCD low–energy theorem of scale invariance and the anomalous $U(1)_A$ Ward identities, one can predict the dispersion of N and Δ in the instanton medium from first principles Second, as we have seen, the instanton medium, obtained with a concrete ansatz for the instanton–instanton and instanton–fermion interaction, exhibits a dispersion of N and Δ in accordance with the general principles. An essential requirement is a fully self–consistent treatment of the instanton medium, based on instanton interactions — any external stabilizing mechanism would violate the renormalization properties.

Our second aim has been to develop a consistent practical prescription to evaluate operators — in particular, gluon operators — in the grand canonical ensemble of instantons with fermions. We have invoked the large–N_c limit, which simplifies the dynamics in two important aspects: First, the relative width of the fluctuations of the total number of pseudoparticles becomes small. Second, in the large–N_c limit one can explicitly construct the effective fermion action for an ensemble with a fixed number of instantons. This enables us to evaluate correlation functions in a two–step procedure: We first integrate over instanton coordinates and the fermion field in a canonical ensemble, and then average the result over fluctuations of $N_\pm$. All fixed–$N_\pm$ correlation functions can be represented as integrals over the effective fermion theory, in which gluon operators are replaced by effective fermion operators. We have shown that in this approach the trace and $U(1)_A$ anomalies are realized at the level of hadronic matrix elements of the operators $\int F_{\mu\nu}^2$ and $\int F_{\mu\nu}\widetilde{F}_{\mu\nu}$. Consequently, the result for the isosinglet axial coupling of the nucleon of [27] may be regarded as a consistent estimate of the nucleon matrix element of the topological charge in the instanton vacuum.

We have established, that our scheme of approximations (variational treatment of instanton interactions, N_c–limit for the fermions, zero mode approximation) provides a framework for evaluating gluonic operators, which is consistent with the basic renormalization properties of QCD. Any attempt to improve these approximations (*e.g.* using more sophisticated trial functions, including explicit correlations between instantons due to fermions, or going beyond the zero–mode approximation for the fermion propagator) should preserve these standards of consistency.

An application of the methods developed here is the calculation of nucleon structure functions, both for leading and non-leading twist. By the operator product expansion of QCD, the moments of the structure functions are related to forward matrix elements of certain local operators which contain gluon fields. These matrix elements can be evaluated using the technique of effective operators developed here. The moments are obtained at the scale set by the instanton background, typically $1/\bar{\rho}$, and have to be put into QCD evolution equations to compare with experimental information at higher momenta. Work in this direction is in progress.

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